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Methods of Integral Transforms for Solving Some Differential Equations of Fractional Order

by

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DEDICATION

For learners everywhere and at all time .

For the detainees and the martyrs who sacrificed their lives for the nation.

For my family; parents , wife and brothers. I also wouldn’t forget my sons (Raghad and Ismail) .

For Al-Azhar University of Gaza which offered me this opportunity for advanced research.

For my classmates and colleagues .

DECLARATION

I declare that this whole work submitted for the degree of Master is the result of my own work, except where otherwise acknowledged in the text, and that this work (or any part of it) has not been submitted for another degree at any other university or institution.

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ABSTRACT

Fractional differential equations, which have derivatives of non-integer order, are very successful in describing natural and physical phenomenon like anomalous kinetics, transport, and chaos. To obtain the exact solutions of these equations, integral transforms are successful used .

In this thesis , we investigate the following items :

1) we discuss the analytical solution of three kinds of the diffusion equation. The standard diffusion equation , the so-called time-fractional diffusion equation with Caputo sense and fractional diffusion equation with Caputo sense and Weyl derivative.

2) we defined the so-called time-fractional telegraph equation with Caputo sense and Weyl derivative then we derive the analytical solution for three basic problems. The whole-space domain and half-space domain problems are solved by applying the Laplace and Fourier transforms in variables t and x , respectively.
The bounded space domain problem is also solved by the spatial Sine transform and temporal Laplace transform, whose solution is given in the form of a series.

3) we defined the so-called time-fractional evolution equation with Caputo sense and Weyl derivative then we derive the fractional Green function to obtain the analytical solutions of the time-fractional wave equation , linearized time-fractional Burgers equation and linear time-fractional KdV equation .

الملخص:

المعادلات التفاضلية الكسرية والتي لها مشتقات من رتبة العدد غير الصحيح هي ناجحة جدا في وصف الظاهرة الطبيعية والفيزيائية مثل الحركة الشاذة والنقل والفوضى , أيضا يمكن الحصول على الحلول الدقيقة لهذه المعادلات باستخدام التحويلات التكاملية والتي تعد ناجحة في الاستخدام .

في هذه الأطروحة ، قمنا بتحقيق البنود التالية :

1. مناقشة الحلول التحليلية لثلاثة أنواع من معادلات الانتشار: المعادلة القياسية للانتشار و ما يسمى معادلة الانتشار الكسرية للزمن مع مشتقة الكابوتو ومعادلة الانتشار الكسرية مع مشتقة الكابوتو والوبيل.
2. تعريف ما يسمى معادلة التلغراف الكسرية للزمن , من ثم اشتقاق الحلول التحليلية لثلاثة مشاكل أساسية هي : حل المشكلات في مجال الفضاء كله ونصف مجال الفضاء من خلال تطبيق تحويلات لابلاس وفوريير في المتغيرات t و x على الترتيب . و حل مشكلة مجال الفضاء المحدود أيضا من قبل تحويلات الجيب و لابلاس و الذي يرد فيه الحل على شكل متسلسلات .
3. تعريف ما يسمى معادلة التطور الكسرية للزمن مع مشتقة الكابوتو و الوبيل , و من ثم اشتقاق معادلة جرين للحصول على الحلول التحليلية لـ : المعادلة التفاضلية الكسرية للموجة الزمنية والمعادلة التفاضلية الكسرية لبرغر للزمن الخطي والمعادلة التفاضلية الكسرية (ك د ف) للزمن الخطي .

CONTENTS

CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

Fractional differential and integral equations have many applications in applied sciences and engineering. These applications appears in gravitation elastic membrane, electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics.

Typical examples of fractional differential and integral equations of the time fractional advection-dispersion equation can be found in [13,18], fractional diffusion equations are illustrated in [22-26], [44,54] and [17] respectively.

The fractional differential and integral equations have been studied and explicit solutions have been achieved by Mainardi, Pagnini and Saxena [24], Langlands [16], Mainardi, Pagnini and Gorenflo [23], Mainardi and Pagnini [25,26], Yu and Zhang [54], Liu, Anh, Turner and Zhang [18], Saxena, Mathai and Haubold [47], Wyss [53], Schneider and Wyss [48] and several other research works can be found in the literature.

In the previous mentioned studies works, the techniques of using integral transforms like Laplace, Fourier and Mellin transforms were used to maintain exact solutions of fractional differential and integral equations explicitly.

1.2 Integral transforms

1.2.1 Laplace integral transform:

The Laplace integral transform [22,23,39] of the function $f(x)$ is defined as

$$\ell\{f(x);p\}=\int_0^\infty f(x)e^{-px}dx=\tilde{f}(p) \qquad ,\mathrm{Re}(p)>0 \tag{1.2.1}$$

where ℓ is the Laplace operator.

The original $f(t)$ can be restored from the Laplace transform $\tilde{f}(p)$ with the help of the inverse Laplace transform

$$f(t)=\ell^{-1}\{\tilde{f}(p),t\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\tilde{f}(p)dp \qquad ,c=\mathrm{Re}(p),\ \ i=\sqrt{-1} \tag{1.2.2}$$

The convolution theorem is defined as :

If $\ell\{f(t), p\} = \tilde{f}(p)$ and $\ell\{g(t), p\} = \tilde{g}(p)$ then

$$\ell\{f(t) * g(t)\} = \ell\{f(t)\}\,\ell\{g(t)\} = \tilde{f}(p)\,\tilde{g}(p) \tag{1.2.3}$$

or, equivalently

$$\ell^{-1}\left\{\tilde{f}(p) * \tilde{g}(p)\right\} = f(t) * g(t) \tag{1.2.4}$$

where $f(t)*g(t)$ is the convolution of $f(t)$ and , $g(t)$ defined by the integral

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau \tag{1.2.5}$$

1.2.2 Fourier integral transform:

The exponential Fourier transform [5,39] of a continuous function $f(t)$ absolutely integrable in $(-\infty, \infty)$ is defined as

$$\Im\{f(t);\kappa\} = \hat{f}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa t} f(t) dt \tag{1.2.6}$$

where $\Im$ is the Fourier operator.

The original $f(t)$ can be obtained from its Fourier inverse transform as

$$f(t) = \Im^{-1}\{\hat{f}(k);t\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\kappa) e^{i\kappa t} d\kappa \tag{1.2.7}$$

The Fourier transform of the convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \tag{1.2.8}$$

is

$$\Im\{f(t) * g(t);\kappa\} = \hat{f}(\kappa)\hat{g}(\kappa) \tag{1.2.9}$$

also, the Fourier transform of the n^{th} derivative of $f(t)$ is

$$\Im\left\{f^{(n)}(t);\kappa\right\} = (i\kappa)^n \hat{f}(\kappa) \tag{1.2.10}$$

1.2.3 Mellin integral transform:

Mellin integral transform [5,39] f^* of a function $f(t)$, in the interval $(0,\infty)$ is defined as

$$f^*(s) = M\{f(t);s\} = \int_0^\infty t^{s-1} f(t) dt, \quad \gamma_1 < \mathrm{Re}(s) < \gamma_2 \tag{1.2.11}$$

and the original $f(t)$ can be obtained from its Mellin transform $f^*(s)$ with the help of the inverse Mellin transform as

$$f(t) = M^{-1}\{f^*(s);t\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-s} f^*(s) ds, \quad 0 < t < \infty \tag{1.2.12}$$

in which $\gamma_1 < \gamma < \gamma_2$.

Also from the definition of the Mellin transform

$$M\left\{t^{-\alpha} f\left(t\right);s\right\} = M\left\{f(t);s-\alpha\right\} = f^*\left(s-\alpha\right) = \int_0^\infty f(t) t^{s-\alpha-1} dt \tag{1.2.13}$$

also

$$M\left\{f\left(t^\gamma\right);s\right\} = \frac{1}{\gamma} f^*\left(\frac{s}{\gamma}\right) = \frac{1}{\gamma}\int_0^\infty f(t)\, t^{\frac{s}{\gamma}-1} \tag{1.2.14}$$

The Mellin transform of the Mellin convolution

$$f(t) * g(t) = \int_0^\infty f(t\tau) g(\tau) d\tau \tag{1.2.15}$$

is

$$M\left\{\int_0^\infty f(t\tau) g(\tau) d\tau; s\right\} = f^*(s) g^*(1-s) \tag{1.2.16}$$

also

$$M\left\{f^{(n)}(t);s\right\} = \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)}\left[f^{(n-k-1)}(t) t^{s-k-1}\right]_0^\infty + \frac{\Gamma(1-s+n)}{\Gamma(1-s)} f^*\left(s-n\right) \tag{1.2.17}$$

If $f(t)$ and $\mathrm{Re}(s)$ turns all substitutions into zero then

$$M\left\{f^{(n)}(t);s\right\} = \frac{\Gamma(1-s+n)}{\Gamma(1-s)} F(s-n) \tag{1.2.18}$$

1.2.4 Finite Fourier sine transform

The finite Fourier sine transform[4,17] of $f(x)$ for $0 < x < L$ is defined by

$$\mathfrak{I}_s(n) = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \tag{1.2.19}$$

Where n is an integer .

The function $f(x)$ is then called the inverse finite Fourier sine transform of $\mathfrak{I}_s(x)$ and is given by

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\mathfrak{I}_s(n)\sin\left(\frac{n\pi x}{L}\right) \tag{1.2.20}$$

1.3 The fractional integrals and derivatives:

1.3.1 Riemann-Liouville fractional integral:

The idea of fractional derivative or fractional integral [5] can be described in different ways. First, we consider a linear non homogeneous n^{th} order ordinary differential equation

$$D^n y = f(x), \quad b \le x \le c \tag{1.3.1}$$

Then $\{1, x, x^2, \dots, x^{n-1}\}$ is a fundamental set of the corresponding homogeneous equation

$$D^n y = 0 \tag{1.3.2}$$

If $f(x)$ is any continuous on $b \le x \le c$, then for any $a \in (b,c)$

$$y(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt \tag{1.3.3}$$

is the unique solution with the initial data

$$y^{(k)}(a) = 0,\ 0 \le k \le n-1 \tag{1.3.4}$$

or, equivalently

$$y = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)dt \tag{1.3.5}$$

Replacing n by α, where $Re(\alpha) > 0$ in the above formula, we obtain the Riemann-Liouville definition of fractional integral that was reported by Liouville in 1832 and by Riemann in 1876 as

$$ {}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \tag{1.3.6}$$

where ${}_\alpha D_x^{-\alpha} = {}_aJ_x^{\alpha}$ is the Riemann-Liouville integral operator.

1.3.2 Riemann-Liouville fractional derivative:

The Riemann-Liouville fractional derivative [5,14,23] is defined as

$$ {}_0D_x{}^{\alpha}f\,(x)=D^nJ^{n-\alpha}f\,(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1} f\,(t)\,dt,\;\; n-1<\alpha\le n \qquad (1.3.7)$$

The formula for the Laplace transform of the Riemann-Liouville fractional derivative is:

$$\int_0^{\infty} e^{-pt}\left\{{}_0D_t^{\alpha} f(t)\right\}dt = p^{\alpha}\tilde{f}(p)-\sum_{k=0}^{n-1} p^k\,{}_0D_t^{\alpha-k-1} f(t)\Big|_{t=0},\; n-1\le\alpha<n \qquad (1.3.8)$$

The formula for the Fourier transform of the Riemann-Liouville fractional derivatives with the lower terminal $a=-\infty$ is

$$\Im\left\{{}_0D_t{}^{\alpha}g\,(t);\kappa\right\}=(-i\,\kappa)^{\alpha-n}\,\Im\left\{g^{(n)}(t);\kappa\right\}=(-i\,\kappa)^{\alpha-n}(-i\,\kappa)^{n}\,\hat{g}\,(\kappa)=(-i\,\kappa)^{\alpha}\,\hat{g}\,(\kappa) \quad (1.3.9)$$

According to the definition of the Riemann-Liouville fractional derivative, we can write

$$ {}_0D_t^{\alpha} f(t)=\frac{d^n}{dt^n}\,{}_0D_t^{-(n-\alpha)} f(t) \qquad (1.3.10)$$

Then the formula for the Mellin transform of the Riemann-Liouville fractional derivative [39] is

$$M\left\{{}_0D_t^{\alpha}f\,(t);s\right\}=\sum_{k=0}^{n-1}\frac{\Gamma(1-s+k)}{\Gamma(1-s)}\left[{}_0D_t^{\alpha-k-1}f\,(t)t^{s-k-1}\right]_0^{\infty}+\frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}f^{*}(s-\alpha) \quad (1.3.11)$$

where $n-1<\alpha\le n$.

Now, If $0<\alpha<1$, then

$$M\left\{{}_0D_t^{\alpha} f(t);s\right\}=\left[{}_0D_t^{\alpha-1} f(t)t^{s-1}\right]_0^{\infty}+\frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\,f^{*}(s-\alpha) \qquad (1.3.12)$$

If the function $f\,(t)$ and $\mathrm{Re}(s)$ turns all substitutions into zero in (1.2.18), then it takes on the simplest form

$$M\left\{{}_0D_t^{\alpha} f(t);s\right\}=\frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\,f^{*}(s-\alpha) \qquad (1.3.13)$$

1.3.3 Caputo fractional derivative:

Caputo [5,14,23] developed the definition of the Riemann-Liouville fractional derivative so he defined the fractional derivative in another way as

$$\frac{d^{\alpha}}{dt^{\alpha}}f(x)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x\frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, & n-1<\alpha\le n\\ \dfrac{d^n f(x)}{dt^n}, & \alpha=n\end{cases} \tag{1.3.14}$$

The Laplace transform of the Caputo derivative is

$$\int_0^{\infty}e^{-pt}\left\{\frac{d^{\alpha}}{dt^{\alpha}}f(t)\right\}dt=p^{\alpha}\tilde{f}(p)-\sum_{k=0}^{n-1}p^{\alpha-k-1}f^{(k)}(0), \quad n-1<\alpha\le n \tag{1.3.15}$$

Note that, when applying Laplace transform to Caputo derivative, integer order initial condition are needed according to the above equation (1.3.15) but comparing to equation (1.3.8), it is noted that fractional order initial condition are needed if the Riemann-liouville derivative is considered .

This means that Caputo derivative is more appropriate in physical and real world applications .

Since integer order initial conditions have physical meaning but fractional order initial conditions are physically meaningless .

The formula for Caputo fractional derivatives with the lower terminal $a=-\infty$ is

$$\Im\left\{\frac{d^{\alpha}}{dt^{\alpha}}g(t);\kappa\right\}=(-i\kappa)^{\alpha-n}\,\Im\left\{g^{(n)}(t);\kappa\right\}=(-i\kappa)^{\alpha-n}(-i\kappa)^{n}\,\hat{g}(\kappa)=(-i\kappa)^{\alpha}\,\hat{g}(\kappa) \tag{1.3.16}$$

which is the same as the Riemann-Liouville fractional derivative formula.

The Mellin transform of the Caputo fractional derivative is

$$M\left\{\frac{d^{\alpha}}{dt^{\alpha}}f(t);s\right\}=\sum_{k=0}^{n-1}\frac{\Gamma(\alpha-m-s)}{\Gamma(1-s)}\left[f^{(k)}(t)t^{s-\alpha+k}\right]_0^{\infty}+\frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}f^{*}(s-\alpha), \quad n-1<\alpha\le n \tag{1.3.17}$$

for $0<\alpha<1$ the pervious formula has the form

$$M\left\{\frac{d^{\alpha}}{dt^{\alpha}}f(t);s\right\}=\frac{\Gamma(\alpha-s)}{\Gamma(1-s)}\left[f(t)t^{s-\alpha}\right]_0^{\infty}+\frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}f^{*}(s-\alpha) \tag{1.3.18}$$

If the function $f(t)$ and $\mathrm{Re}(s)$ turns all substitutions into zero in (1.2.18), then it takes on the simplest form

$$M\left\{\frac{d^{\alpha}}{dt^{\alpha}}f(t);s\right\}=\frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}f^{*}(s-\alpha) \tag{1.3.19}$$

The Mellin transform is related to the Laplace transform [44] by the relation

$$f^*(s) = \frac{1}{\Gamma(1-s)}\int_0^\infty p^{-s}\tilde{f}(p)dp \tag{1.3.20}$$

also, the Mellin transform is related to the Fourier transform [44] by the relation

$$M\left\{\hat{f}(\kappa);s\right\} = 2\Gamma(s)\cos\left(\frac{\pi s}{2}\right)M\{f(x);1-s\} = 2\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\int_0^\infty x^{-s}f(x)dx \tag{1.3.21}$$

1.3.4 Weyl fractional derivative :

The Weyl fractional derivative [30,31,46] defined by

$${}_{-\infty}D_x^{\mu}f(t) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dt^n}\int_{-\infty}^{t}(t-u)^{n-\mu-1}f(u)du \tag{1.3.22}$$

where $n = [\mu]$ is an integer part of $\mu > 0$

Its Fourier transform [31] is

$$\Im\left\{{}_{-\infty}D_x^{\mu}f(x);\kappa\right\} = (i\kappa)^{\mu}f(\kappa) \tag{1.3.23}$$

By suppressing the imaginary part of the above formula, one can write the Fourier transform of Weyl derivative as [31]

.

$$\Im\left\{{}_{-\infty}D_x^{\mu}f(x);\kappa\right\} = -|\kappa|^{\mu}f(\kappa) \tag{1.3.24}$$

1.4 Some special functions

1.4.1 Gamma function:

The gamma function [39,41] $\Gamma(z)$ is defined as

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \quad ,\mathrm{Re}\, z > 0 \tag{1.4.1}$$

One of the basic properties of the gamma function is

$$\Gamma(z+1) = z\Gamma(z) \tag{1.4.2}$$

which we can be proved by integrating by parts

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^{z}dt = \left[-e^{-t}t^{z}\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z) \tag{1.4.3}$$

Obviously, $\Gamma(1) = 1$, and by using (1.4.2) we obtain the following values for $\Gamma(z)$ when $z = 1,2,3,\ldots$

$$
\begin{aligned}
&\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1! \\
&\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1! = 2! \\
&\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2! = 3! \\
&\vdots \\
&\Gamma(n+1) = n\cdot\Gamma(n) = n\cdot(n-1)! = n!
\end{aligned}
\tag{1.4.4}
$$

also, there is a useful relationship of the gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{1.4.5}$$

and from the last relation, we have the result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{1.4.6}$$

1.4.2 Mittag-Leffler function

The Mittag-Leffler function [26,39,54] of one-parameter is denoted by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{1.4.7}$$

which was introduced by G.M.Mittag-Leffler and studied also by A.Wiman. A two-parameter function of the Mittag-Leffler type is defined by the series expansion as

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ (\alpha>0,\ \beta>0) \tag{1.4.8}$$

it follows that

$$E_{1,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z \tag{1.4.9}$$

also

$$E_{1,2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty}\frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!} = \frac{e^z-1}{z} \tag{1.4.10}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty}\frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!} = \frac{e^z-1-z}{z^2} \tag{1.4.11}$$

In general

$$E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2}\frac{z^k}{k!}\right\} \tag{1.4.12}$$

note that

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}=E_{\alpha}(z) \tag{1.4.13}$$

The inverse Laplace transform of some useful formulas produce Mittag-Leffler function of one and two parameters [32,33] are

$$\ell^{-1}\left\{\frac{p^{\alpha-\beta}}{p^{\alpha}\pm c};t\right\}=t^{\beta-1}E_{\alpha,\beta}\left(\mp ct^{\alpha}\right),\quad c>0 \tag{1.4.14}$$

$$\ell^{-1}\left\{\frac{p^{2\alpha-1}+ap^{\alpha-1}}{p^{2\alpha}+ap^{\alpha}+b};t\right\}=\frac{1}{\sqrt{(a^2-4b)}}\left[(\lambda+a)E_{\alpha}(\lambda t^{\alpha})-(\mu+a)E_{\alpha}(\mu t^{\alpha})\right] \tag{1.4.15}$$

and

$$\ell^{-1}\left\{\frac{1}{p^{2\alpha}+ap^{\alpha}+b};t\right\}=\frac{t^{\alpha-1}}{\sqrt{(a^2-4b)}}\left[E_{\alpha,\alpha}(\lambda t^{\alpha})-E_{\alpha,\alpha}(\mu t^{\alpha})\right] \tag{1.4.16}$$

where $a^2-4b>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(p)>0$

λ and μ are the real and distinct roots of the quadratic equation $x^2+ax+b=0$, namely

$$\lambda=\frac{1}{2}(-a+\sqrt{(a^2-4b)})\quad\text{and}\quad\mu=\frac{1}{2}(-a-\sqrt{(a^2-4b)})\,.$$

1.4.3 The Wright function

The Wright function [26,40] of two parameters α and β is defined by the series representation

$$W(\alpha,\beta,z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)j\,!} \tag{1.4.17}$$

where $\alpha\in\mathbb{R}$, $\beta\in\mathbb{C}$. this is an entire function for $\alpha>-1$

The Mellin transform for the Wright function [31] $W(-\alpha,0,-t)$, $t>0$ is given by

$$\int_0^{\infty}t^{s-1}W(-\alpha,0,-t)dt=\frac{\Gamma(s)}{\Gamma(\alpha s)} \tag{1.4.18}$$

The generalized Wright function [20] is defined by the series expression

$$W_{(\alpha,a)(\beta,b)}(z)=\sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+a)\Gamma(\beta j+b)} \tag{1.4.19}$$

where $\alpha,\beta\in\mathbb{R}$ and $a,b\in\mathbb{C}$.

This function is an entire function for $0<\alpha<\beta$ and $a,b\in\mathbb{C}$

1.4.4 Fox's *H*-function

An H-function is defined in terms of a Mellin-Barnes type integral as follows [24,25,47,50,54]:

$$H_{p,q}^{m,n}(z)=H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,\alpha_1),...,(a_p,...,\alpha_p)\\(b_1,\beta_1),...,(b_q,\beta_q)\end{matrix}\right]=\frac{1}{2\pi i}\int_L H_{p,q}^{*m,n}(s)z^{-s}ds \tag{1.4.20}$$

where m,n,p and q are nonnegative integers such that $0\le n\le p$, $1\le m\le q$ and empty products are interpreted as unity. The parameters $\alpha_1,...,\alpha_p$ and $\beta_1,...,\beta_q$ are positive real numbers, where as $a_1,\ldots,a_p$ and $b_1,\ldots b_q$ are complex numbers. The Fox or *H*-function is characterized by its Mellin transform:

$$H_{p,q}^{*m,n}(z)=H_{p,q}^{*m,n}\left[z\middle|\begin{matrix}(a_j,\alpha_j)_{j=1,...,p}\\(b_i,\beta_i)_{i=1,...,q}\end{matrix}\right](s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\prod_{j=1}^{n}\left(1-a_j-\alpha_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\alpha_j s\right)} \tag{1.4.21}$$

For the conditions of existence of the H-function, one can refer to ref. [50].

1.4.5 The Dirac delta function:

The Dirac delta function [42] and its derivative play a useful role in the solution of initial boundary value problems. Consider the function having the following property

$$\delta_\varepsilon(t)=\begin{cases}1/2\varepsilon, & |t|<\varepsilon\\ 0, & |t|>\varepsilon\end{cases} \tag{1.4.22}$$

Thus

$$\int_{-\infty}^{\infty}\delta_\varepsilon(t)dt=\int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}dt=1 \tag{1.4.23}$$

Let $f(t)$ be any function which is integrable in the interval $(-\varepsilon,\varepsilon)$, then

$$\int_{-\infty}^{\infty} f(t)\delta_{\varepsilon}(t)dt = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t)dt = f(\xi), \quad -\varepsilon < \xi < \varepsilon \tag{1.4.24}$$

The Dirac delta function defined as

$$\delta(t) = \lim_{\varepsilon \to 0}\delta_{\varepsilon}(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{1.4.25}$$

The important properties of dirac delta function are:

Property1

$$\int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{1.4.26}$$

Property2

For any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a). \tag{1.4.27}$$

From the last formula we can obtain the Fourier transform of the Dirac delta function as

$$\Im\{\delta(t-a);\kappa\} = \int_{-\infty}^{\infty} e^{i\kappa t}\delta(t-a)dt = e^{i\kappa a} \tag{1.4.28}$$

when $a = 0$, obtain formal result

$$\Im\{\delta(t);\kappa\} = \hat{\delta}(\kappa) = 1 \tag{1.4.29}$$

The Laplace transform of the delta function obtain by

$$\ell\{\delta(t-a);p\} = \int_{0}^{+\infty} \delta(t-a)e^{-pt}dt = e^{-ap} \tag{1.4.30}$$

when $a = 0$, obtain the formula result

$$\ell\{\delta(t);p\} = \int_{0}^{+\infty} \delta(t)e^{-pt}dt = 1 \tag{1.4.31}$$

CHAPTER 2

Diffusion Equations of Fractional Order

2.1 Introduction

Anomalous diffusion has been known since Richardson's treatise on turbulent diffusion in 1926 [43]. Many researchers have conducted their work in this area, and anomalous diffusion in the presence or absence of an external velocity or force field has been modeled in numerous ways, in which one of the strongest tools is fractional calculus .

The applications of this theory have grown rapidly in the past few decades, which has attracted fairly broad research activities. A topical review given by Metzler and Klafter summarized the development of anomalous cases in various fields [31]. However, the application of fractional calculus in quantum mechanics is at its very beginning [51].

The Schrödinger equation has the mathematical appearance of diffusion equation and can be derived by considering probability distributions. Feynman and Hibbs used a Gaussian probability distribution in the space of all possible paths, for a quantum mechanical particle, to derive the Schrödinger equation [8]. The main physical purpose for adopting and investigating diffusion equations of fractional order is to describe phenomena of anomalous diffusion usually met in transport processes through complex and/or disordered systems including fractal media. In this respect, in recent years interesting reviews, have appeared in [30,31,38,55], All the related models of random walk turn out to be beyond the classical Brownian motion, which is known to provide the microscopic foundation of the standard diffusion [15,49].

The diffusion-like equations containing fractional derivatives in time and/or in space are usually adopted to model phenomena of anomalous transport in physics.

2.2 The Standard Diffusion Equation

The standard diffusion equation [23,25] is known to be

$$\frac{\partial}{\partial t}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t), \qquad t>0, \quad x\in\mathbb{R} \tag{2.2.1}$$

with $u(x,t)$ as a field variable , subject to the initial conditions

$$\begin{aligned} &u(x,0^+)=f(x) \\ &u(x,t)\to 0 \quad as \quad |x|\to\infty \end{aligned} \tag{2.2.2}$$

where $f(x)$ denotes a given ordinary or generalized function defined on $\mathbb{R}$, which we assume to be Fourier transformable in ordinary or generalized sense, respectively.

We assume to work in a suitable space of generalized functions where it is possible to deal freely with Delta functions, integral transforms of Fourier, Laplace and Mellin type, and fractional integrals and derivatives .

It is well known that the fundamental solution (or Green function) of Equation (2.2.1), i.e. the solution subjected to the initial condition $u(x,0^{+})=f(x)=\delta(x)$ is

$$u(x,t)=\frac{1}{2\sqrt{\pi}}t^{-1/2}e^{-x^{2}/(4t)} \tag{2.2.3}$$

That evolves in time with second moment growing linearly with time

$$\mu_2(t)=\int_{-\infty}^{\infty}x^2u(x,t)dx=2t \tag{2.2.4}$$

It is known that the Cauchy problem of (2.2.1), subject to (2.2.2) is equivalent to the integro-differential equation

$$u(x,t)=f(x)+\int_0^t\left[\frac{\partial^2}{\partial x^2}u(x,\tau)\right]d\tau \tag{2.2.5}$$

where the initial condition is incorporated .

2.3 Time-Fractional Diffusion Equation

By replacing in the standard diffusion equation the first order time derivative by fractional derivative of order $\alpha\in(0,1]$ and we can write

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t) \quad , \quad x\in\Re \quad , \quad t>0 \quad , \quad 0<\alpha\le 1 \tag{2.3.1}$$

where, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is fractional derivative of the Caputo sense of order α ,

$0<\alpha\le 1$ and $u(x,t)$ as afield variable , subject to the initial conditions

$$u(x,0^{+})=\delta(x)$$

$$u(x,t)\to 0 \quad as \quad |x|\to\infty \tag{2.3.2}$$

The Laplace transform of (2.3.1) with respect to t is

$$p^{\alpha}\tilde{u}(x,p)-p^{\alpha-1}u(x,0)=\frac{\partial^2}{\partial x^2}\tilde{u}(x,p), \quad x\in\Re, \quad 0<\alpha\le 1 \tag{2.3.3}$$

Furthermore the Fourier transform of (2.3.3) with respect to x is

$$p^{\alpha}\hat{\tilde{u}}(\kappa,p)-p^{\alpha-1}\hat{u}(\kappa,0)=-\kappa^2\hat{\tilde{u}}(\kappa,p), \quad 0<\alpha\le 1 \tag{2.3.4}$$

and this equation leads to

$$\hat{\tilde{u}}(\kappa,p)=\frac{p^{\alpha-1}\hat{u}\left(\kappa,0\right)}{p^{\alpha}+\kappa^{2}} \tag{2.3.5}$$

but using eq.(1.4.29) $\hat{u}(\kappa,0)=\hat{\delta}(\kappa)=1$

Then Eq. (2.3.5) yields

$$\hat{\tilde{u}}(\kappa,p)=\frac{p^{\alpha-1}}{p^{\alpha}+\kappa^{2}} \tag{2.3.6}$$

to determine the Green function [23,25] (that is expected to be symmetric in *x*) in the space-time domain, we can follow two alternative strategies :

(S1) invert the Fourier transform getting $\tilde{u}(x,p)$ and then invert this Laplace transform.

(S2) invert the Laplace transform getting $\hat{u}(\kappa,t)$ and then invert this Fourier transform.

(S1): Recalling the inversion Fourier transform

$$\Im^{-1}\left\{\frac{a}{b+\kappa^{2}};x\right\}=\frac{a}{2\sqrt{b}}e^{-|x|\sqrt{b}} \quad , \quad b>0 \tag{2.3.7}$$

and setting $a=p^{\alpha-1}, \quad b=p^{\alpha}$, we get

$$\tilde{u}\left(x,p\right)=\frac{p^{\frac{\alpha}{2}-1}}{2}e^{-|x|p^{\alpha/2}}, \quad 0<\alpha\leq 1 \tag{2.3.8}$$

The strategy (S1) has been followed by Mainardi [27] to obtain

$$u\left(x,t\right)=\frac{1}{2}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}e^{-|x|s^{\alpha/2}}\frac{ds}{s^{1-\frac{\alpha}{2}}}=\frac{1}{2}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st-|x|s^{\alpha/2}}\frac{ds}{s^{1-\frac{\alpha}{2}}} \tag{2.3.9}$$

Let $\sigma=st, \quad then \quad , \quad d\sigma=tds$, and then Eq.(2.3.9) yields

$$u\left(x,t\right)=\frac{t^{-\frac{\alpha}{2}}}{2}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\sigma-X\sigma^{\alpha/2}}\frac{d\sigma}{\sigma^{1-\frac{\alpha}{2}}} \tag{2.3.10}$$

where $X:=\frac{|x|}{t^{\frac{\alpha}{2}}}$

we can write Eq.(2.3.10) as

$$u\left(x,t\right)=t^{-\frac{\alpha}{2}}U\left(\frac{|x|}{t^{\frac{\alpha}{2}}}\right), \quad -\infty<x<\infty, \quad t>0 \tag{2.3.11}$$

where $U(X)$ is the Green function and $U(x):=u(x,1)$

Also we can denote the last Green function in terms of a Wright function [26] of the second type as

$$U(x) = \frac{1}{2} M_{\alpha/2}\left(|x|\right) \tag{2.3.13}$$

where the *M* function of order γ defined as

$$M_{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma\left[-\gamma k + (1-\gamma)\right]} \qquad ,0 < \gamma < 1 \tag{2.3.14}$$

the *M* function of order $\alpha/2$ has been introduced and investigated in [26,28,40] as

$$M_{\alpha/2}\left(|x|\right) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\sigma - X\sigma^{\alpha/2}} \frac{d\sigma}{\sigma^{1-\frac{\alpha}{2}}} \tag{2.3.15}$$

and for any $\alpha \in (0,2)$ the function $M_{\alpha/2}(x)$ can be defined by a power series as

$$\begin{aligned} M_{\alpha/2}(x) &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma\left[-\alpha k/2 + (1-\alpha/2)\right]} \\ &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma\left[1 - \frac{\alpha(k+1)}{2}\right]} \\ &= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \Gamma\left[\left(\alpha(k+1)\right)/2\right] \sin\left[\pi\alpha(k+1)/2\right] \end{aligned} \tag{2.3.16}$$

Now, we will take another side

(S2): Recalling the inversion Laplace transform [39]

$$\ell^{-1}\left\{\frac{p^{\alpha-1}}{p^{\alpha}+c};t\right\} = E_{\alpha}\left(-ct^{\alpha}\right), \quad c > 0 \tag{2.3.17}$$

and setting $c = \kappa^2$ we get

$$\hat{u}(\kappa,t) = E_{\alpha}\left(-\kappa^2 t^{\alpha}\right), \quad 0 < \alpha \le 1 \tag{2.3.18}$$

where E_{α} denotes the Mittag-Leffler function of one parameter

The strategy (S2) has been followed by Gorenflo et al. [10] and by Mainardi et al. [28] by the inverse Fourier cosine transform defined as

$$\Im_c^{-1}\left\{\hat{f}_c(\kappa);x\right\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(\kappa x) \hat{f}_c(\kappa) d\kappa \tag{2.3.19}$$

where $\Im_c^{-1}$ is the inverse Fourier cosine transform operator to obtain the exact solution of the diffusion equation as :

$$u(x,t)=\frac{1}{\pi}\int_0^\infty \cos(\kappa x)E_\alpha(-\kappa^2 t)d\kappa \tag{2.3.20}$$

and we can obtain the Green function as

$$U(x)=u(x,1)=\frac{1}{\pi}\int_0^\infty \cos(\kappa x)E_\alpha(-\kappa^2)d\kappa=\frac{1}{2x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(1-p)}{\Gamma(1-\alpha p/2)}x^p dp \tag{2.3.21}$$

with $0<\gamma<1 \quad and \quad x>0$

In conclusion, we may represent the solution $U(x)$ given in (2.3.13) and (2.3.19) in terms of a Fox *H*-function as follows

$$U(x)=\frac{1}{2}H_{1,1}^{1,0}\left[x\left|\begin{matrix}(1-\alpha/2,\alpha/2)\\(0,1)\end{matrix}\right.\right] \tag{2.3.22}$$

Furthermore the moments (of even order) of $u(x,t)$ are

$$\mu_{2n}(t):=\int_{-\infty}^{\infty}x^{2n}u(x,t)dx=\frac{\Gamma(2n+1)}{\Gamma(\alpha n+1)}t^{\alpha n},\quad n=0,1,2,\dots \quad t>0 \tag{2.3.23}$$

Now, from the last formula the second moment denotes as

$$\mu_2(t)=\frac{\Gamma(3)}{\Gamma(\alpha+1)}t^\alpha=2\frac{t^\alpha}{\Gamma(\alpha+1)},\quad 0<\alpha\le 1 \tag{2.3.24}$$

since

$$\hat{u}(\kappa,t)=\sum_{n=0}^{\infty}\frac{(-\kappa^2 t^\alpha)^n}{\Gamma(\alpha n+1)}=1-\frac{-\kappa^2 t^\alpha}{\Gamma(\alpha+1)}+\sum_{n=2}^{\infty}\frac{(-1)^n(\kappa^2 t^\alpha)^n}{\Gamma(\alpha n+1)} \tag{2.3.25}$$

then

$$\frac{\partial\hat{u}(\kappa,t)}{\partial\kappa}=\frac{-2\kappa t^\alpha}{\Gamma(\alpha+1)}+\sum_{n=2}^{\infty}\frac{(-1)^n 2n\kappa^{2n-1}t^{\alpha n}}{\Gamma(\alpha n+1)} \tag{2.3.26}$$

$$\frac{\partial^2\hat{u}(\kappa,t)}{\partial\kappa^2}=\frac{-2t^\alpha}{\Gamma(\alpha+1)}+\sum_{n=2}^{\infty}\frac{(-1)^n 2n(2n-1)\kappa^{2n-2}t^{\alpha n}}{\Gamma(\alpha n+1)} \tag{2.3.27}$$

hence

$$\left.\frac{\partial^2\hat{u}(\kappa,t)}{\partial\kappa^2}\right|_{\kappa=0}=-\frac{2t^\alpha}{\Gamma(\alpha+1)} \tag{2.3.28}$$

So we see that

$$\mu_2(t) = 2\frac{t^{\alpha}}{\Gamma(\alpha+1)} = -\frac{\partial^2 \hat{u}(\kappa=0,t)}{\partial \kappa^2}, \quad 0<\alpha\le 1 \tag{2.3.29}$$

2.4 Fractional Diffusion Equation

By replacing, in the time-fractional diffusion equation, the second order space derivative by Weyl fractional derivative of order $\beta+1$, $\beta\in(0,1]$ and we can write

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = {}_{-\infty}D_x^{\beta+1} u(x,t) \quad , \quad t>0 \quad , \quad x\in\Re \tag{2.4.1}$$

where, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is fractional derivative of the Caputo sense of order α, ${}_{-\infty}D_x^{\beta+1}$ is Weyl fractional derivative and $0<\alpha\le 1$, $0<\beta\le 1$.

$u(x,t)$ as a field variable , subject to the initial conditions

$$\begin{aligned} &u(x,0^{+}) = f(x) \\ &u(x,t)\to 0 \quad as \quad |x|\to\infty \end{aligned} \tag{2.4.2}$$

The Laplace transform of (2.4.1) with respect to t is

$$p^{\alpha}\tilde{u}(x,p) - p^{\alpha-1}u(x,0) = {}_{-\infty}D_x^{\beta+1}\tilde{u}(x,p), \quad x\in\Re, \tag{2.4.3}$$

Furthermore the Fourier transform of (2.4.3) with respect to x is

$$p^{\alpha}\hat{\tilde{u}}(\kappa,p) - p^{\alpha-1}\hat{u}(\kappa,0) = (i\kappa)^{\beta+1}\hat{\tilde{u}}(\kappa,p) \tag{2.4.4}$$

and this equation leads to

$$\hat{\tilde{u}}(\kappa,p) = \frac{p^{\alpha-1}\hat{u}(\kappa,0)}{p^{\alpha}-(i\kappa)^{\beta+1}}, \quad 0<\alpha\le 1, \quad 0<\beta\le 1 \tag{2.4.5}$$

but since as in (2.4.2)

$$\hat{u}(\kappa,0) = f(\kappa) , \tag{2.4.6}$$

then Eq. (2.4.5) yields

$$\hat{\tilde{u}}(\kappa,p) = G(\kappa,p) f(\kappa) \tag{2.4.7}$$

where $G(\kappa,p) = \dfrac{p^{\alpha-1}}{p^{\alpha}-(i\kappa)^{\beta+1}}$, (2.4.8)

is the Fourier and Laplace transform of the Green function for the fractional diffusion equation (2.4.1).

Recalling the inverse Laplace transform [39]

$$\ell^{-1}\left\{\frac{p^{\alpha-1}}{p^{\alpha}+c};t\right\}=E_{\alpha}\left(-ct^{\alpha}\right),\quad c>0 \tag{2.4.9}$$

and setting $c=(i\kappa)^{\beta+1}$ we get

$$G\left(\kappa,t\right)=E_{\alpha}\left((i\kappa)^{\beta+1}t^{\alpha}\right),\quad 0<\alpha\leq 1,\quad 0<\beta\leq 1 \tag{2.4.10}$$

where E_{α} denotes the Mittag-Leffler function of one parameter hence we can write Eq.(2.4.7) in the form

$$u(\kappa,t)=G(\kappa,t)f(\kappa) \tag{2.4.11}$$

Now, the solution of the fractional diffusion equation (2.4.1) can be found by taking the invers Fourier transform of the Eq.(2.4.11) to obtain

$$u(x,t)=\int_{-\infty}^{+\infty}G(x-y,t)f(y)dy \tag{2.4.12}$$

2.4.1 Special case :

If $\beta=1$, in Eq.(2.4.1) with the initial and boundary condition (2.4.2) then we obtain

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u\left(x,t\right)={}_{-\infty}D_{x}^{2}u\left(x,t\right),\quad t>0,\quad x\in\Re,\quad 0<\alpha<1 \tag{2.4.13}$$

The solution of the diffusion equation (2.5.1) is the same solution obtained in section (2.3) i.e.

$$u\left(x,t\right)=\frac{t^{-\frac{\alpha}{2}}}{2}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\sigma-\frac{|x|}{t^{\frac{\alpha}{2}}}\sigma^{\alpha/2}}\frac{d\sigma}{\sigma^{1-\frac{\alpha}{2}}}$$

CHAPTER 3

Analytical Solution For The Time – Fractional Telegraph Equation "TFTE"

3.1 Introduction

The fractional telegraph equation has recently been considered by many authors. Cascaval et al. [2] discussed the time-fractional telegraph equations, dealing with wellposedness and presenting a study involving asymptotic by using the Riemann-Liouville approach. Orsingher and Beghin [36] discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations. Chen et al. [3] also discussed and derived the solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separating of variables. Orsingher and Zhao [37] considered the space-fractional telegraph equations, obtaining the Fourier transform of its fundamental solution and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation. Momani [34] discussed analytic and approximate solutions of the space- and time-fractional telegraph differential equations by means of the so-called Adomian decomposition method. Camargo et al. [9] discussed the so-called general space-time fractional telegraph equations by the methods of differential and integral calculus, obtaining the solution by means of the Laplace and Fourier transforms in variables *t* and *x*, respectively .

In this chapter , we consider the following time – fractional telegraph equation "TFTE"

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)+2a\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=d\ {}_{-\infty}D_x^{\beta+1}u(x,t)+f(x,t),\quad t\in\Re^{+} \tag{3.1.1}$$

where a,d are positive constant, $\frac{1}{2}<\alpha\le 1$ and $0<\beta\le 1$

$\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is fractional derivative of the Caputo sense of order α ,

${}_{-\infty}D_x^{\beta+1}$ is Weyl fractional derivative .

For the "TFTE" (3.1.1), we will consider three basic problems with the following three kinds of initial and boundary conditions, respectively.

(i) ***Problem 1***

"TFTE" in a whole – space domain

$$\begin{aligned} &u(x,0)=\phi(x), \qquad \frac{\partial}{\partial t}u(x,0)=0, \qquad x\in\Re \\ &u(\pm\infty,t)=0, \qquad t>0 \end{aligned} \tag{3.1.2}$$

(ii) ***Problem 2***

"TFTE" in a half – space domain

$$u(x,0)=\frac{\partial}{\partial t}u(x,0)=0, \qquad x\in\Re^{+}$$
$$u(0,t)=g(t), \qquad u(+\infty,t)=0, \quad t>0 \tag{3.1.3}$$

(iii) ***Problem 3***

"TFTE" in abounded – space domain

$$u(x,0)=\phi(x), \quad \frac{\partial}{\partial t}u(x,0)=\varphi(x), \quad 0<x\le L$$
$$u(0,t)=u(L,t)=0, \qquad t>0 \tag{3.1.4}$$

here we set $f(x,t)=0$ in (3.1.1).

3.2 The Solution of "TFTE" in a Whole – Space Domain

By applying Fourier transform of the Weyl derivative (1.3.24) with respect to x, and Laplace transform of the Caputo derivative (1.3.15) with respect to t, in (3.1.1), respectively and use the initial and boundary condition (3.1.2) , we obtain the following non- homogeneous differential equation

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(\kappa,t)+2a\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(\kappa,t)=-d\,|\kappa|^{\beta+1}u(\kappa,t)+f(\kappa,t) \tag{3.2.1}$$

hence

$$p^{2\alpha}u(\kappa,p)-p^{2\alpha-1}\phi(\kappa)+2ap^{\alpha}u(\kappa,p)-2ap^{\alpha-1}\phi(\kappa)=-d\,|\kappa|^{\beta+1}u(\kappa,p)+f(\kappa,p) \tag{3.2.2}$$

We can write $u(\kappa,p)$ explicitly as

$$u(\kappa,p)=\frac{\left(p^{2\alpha-1}+2ap^{\alpha-1}\right)\phi(\kappa)}{p^{2\alpha}+2ap^{\alpha}+d\,|\kappa|^{\beta+1}}+\frac{1}{p^{2\alpha}+2ap^{\alpha}+d\,|\kappa|^{\beta+1}}f(\kappa,p) \tag{3.2.3}$$

hence

$$u(\kappa,p)=G_1(\kappa,p)\phi(\kappa)+G_2(\kappa,p)f(\kappa,p) \tag{3.2.4}$$

where

$$G_1(\kappa,p)=\frac{p^{2\alpha-1}+2ap^{\alpha-1}}{p^{2\alpha}+2ap^{\alpha}+d\,|\kappa|^{\beta+1}} \tag{3.2.5}$$

and

$$G_2(\kappa, p) = \frac{1}{p^{2\alpha} + 2ap^{\alpha} + d\,|\kappa|^{\beta+1}} \tag{3.2.6}$$

Now, making use of the formulas given by (1.4.15) and (1.4.16) we can invert the Laplace transform in (3.2.5) and (3.2.6), and we obtain

$$G_1(\kappa,t) = \frac{1}{2\sqrt{(a^2 - d\,|\kappa|^{\beta+1})}}\left[(\lambda+2a)E_{\alpha}(\lambda t^{\alpha}) - (\mu+2a)E_{\alpha}(\mu t^{\alpha})\right] \tag{3.2.7}$$

and

$$G_2(\kappa,t) = \frac{t^{\alpha-1}}{2\sqrt{(a^2 - d\,|\kappa|^{\beta+1})}}\left[E_{\alpha,\alpha}(\lambda t^{\alpha}) - E_{\alpha,\alpha}(\mu t^{\alpha})\right] \tag{3.2.8}$$

where $\lambda = -a + \sqrt{a^2 - d\,|\kappa|^{\beta+1}}$ and $\mu = -a - \sqrt{a^2 - d\,|\kappa|^{\beta+1}}$ (3.2.9)

hence we can write Eq.(3.2.4) by Eq.(3.2.7) and Eq.(3.2.8) to obtain

$$u(\kappa,t) = G_1(\kappa,t)\phi(\kappa) + G_2(\kappa,t)f(\kappa,p) \tag{3.2.10}$$

Now; we invert the Fourier transform in (3.2.10) to obtain

$$u(x,t) = \int_{-\infty}^{+\infty} G_1(x-y,t)\phi(y)dy + \int_{-\infty}^{+\infty}\int_{0}^{t} G_2(x-y,t-\tau)f(y,\tau)d\tau dy, \tag{3.2.11}$$

where $G_1(x,t)$, $G_2(x,t)$ are the corresponding Green functions or fundamental solutions obtained when $\phi(x) = \delta(x),\ f(x) = 0$ and $\phi(x) = 0,\ f(x,t) = \delta(x)\delta(t)$ respectively, which is characterized by (3.2.7) or (3.2.8) .

To more explain the previous paragraph :

(i) when $\phi(x) = \delta(x),\ f(x) = 0$ in the initial and boundary condition (3.1.2)

then Eq.(3.2.10) can be written as

$$u(\kappa,t) = G_1(\kappa,t)\delta(\kappa) \tag{3.2.12}$$

but since as in (1.4.29) $\Im\{\delta(x),\kappa\} = 1$, then we can write Eq.(3.2.12) as

$$u(\kappa,t) = G_1(\kappa,t) \tag{3.2.13}$$

Now, taking the inverse Fourier transform of Eq.(3.2.13) obtain

$$u(x,t) = G_1(x,t) = \frac{1}{2\sqrt{(a^2 - d\,|x|^{\beta+1})}}\left[(\lambda+2a)E_{\alpha}(\lambda t^{\alpha}) - (\mu+2a)E_{\alpha}(\mu t^{\alpha})\right] \tag{3.2.14}$$

(ii) when $\phi(x)=0,\ f(x,t)=\delta(x)\delta(t)$ in the initial and boundary condition (3.1.2) then Eq.(3.2.10) can be written as

$$u(\kappa,t)=G_2(\kappa,t)\delta(\kappa)\delta(p) \qquad (3.2.15)$$

but since from (1.4.29), $\Im\{\delta(x),\kappa\}=1$ and (1.4.32), $\ell\{\delta(t);p\}=1$ then we can write Eq.(3.2.15) as

$$u(\kappa,t)=G_2(\kappa,t) \qquad (3.2.16)$$

Now, taking the inverse Fourier transform of Eq.(3.2.16), we obtain

$$u(x,t)=G_2(x,t)=\frac{t^{\alpha-1}}{2\sqrt{(a^2-d\,|x|^{\beta+1})}}\left[E_{\alpha,\alpha}(\lambda t^{\alpha})-E_{\alpha,\alpha}(\mu t^{\alpha})\right] \qquad (3.2.17)$$

To express the Green function, we recall two Laplace transform pairs [7]

$$F_1^{\gamma}(ct)=t^{-\gamma}M_{\gamma}(ct)\stackrel{p}{\longleftrightarrow}p^{\gamma-1}e^{-cp^{\gamma}} \qquad (3.2.18)$$

$$F_2^{\gamma}(ct)=cw_{\gamma}(ct)\stackrel{p}{\longleftrightarrow}e^{-(p/c)^{\gamma}} \qquad (3.2.19)$$

where M_{γ} denotes the so-called M-function (of the Wright type) of order γ which is defined as [28] :

$$M_{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma[-\gamma n+(1-\gamma)]} \quad , \quad 0<\gamma<1 \qquad (3.2.20)$$

$w_{\gamma}(0<\gamma<1)$ denotes the one-side stable (or levy) probability density which can be explicity expressed by Fox's H-function [48].

$$w_{\gamma}(t)=\gamma^{-1}t^{-2}H_{1,1}^{1,1}\left[t^{-1}\left|\begin{matrix}(-1,1)\\ (\frac{-1}{\gamma},\frac{1}{\gamma})\end{matrix}\right.\right] \qquad (3.2.21)$$

Then the Laplace – Fourier transform of the green function (3.2.5) can be rewritten in integral form :

$$G_1(\kappa,p)=(p^{2\alpha-1}+2ap^{\alpha-1})\int_0^{+\infty}e^{-u(p^{2\alpha}+2ap^{\alpha}+d|\kappa|^{\beta+1})}du$$

$$=\int_0^{+\infty}(p^{2\alpha-1}e^{-up^{2\alpha}})e^{-2aup^{\alpha}}e^{-du|\kappa|^{\beta+1}}du+2a\int_0^{+\infty}(p^{\alpha-1}e^{-2aup^{\alpha}})e^{-up^{2\alpha}}e^{-du|\kappa|^{\beta+1}}du$$

$$= \int_0^{+\infty} \ell\left\{F_1^{(2\alpha)}(ut)\right\}\ell\left\{F_2^{\alpha}\left[(2au)^{-1/\alpha}t\right]\right\}e^{-du|\kappa|^{\beta+1}}du + 2a\int_0^{+\infty} \ell\left\{F_1^{\alpha}(ut)\right\}\ell\left\{F_2^{(2\alpha)}\left[(u^{-1/2\alpha})t\right]\right\}e^{-du|\kappa|^{\beta+1}}du$$

$$= \int_0^{+\infty} \ell\left\{F_1^{(2\alpha)}(ut)*F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}e^{-du|\kappa|^{\beta+1}}du + 2a\int_0^{+\infty} \ell\left\{F_1^{\alpha}(ut)*F_2^{(2\alpha)}\left[(u^{-1/2\alpha})t\right]\right\}e^{-du|\kappa|^{\beta+1}}du$$

Now, we invert the Laplace and Fourier transform to obtain the relation

$$G_1(x,t) = \int_0^{+\infty} \left\{F_1^{(2\alpha)}(ut)*F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-idux|\kappa|^{\beta+2}}d\kappa\right)du$$

$$+2a\int_0^{+\infty} \left\{F_1^{\alpha}(ut)*F_2^{(2\alpha)}\left[(u^{-1/2\alpha})t\right]\right\}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-idux|\kappa|^{\beta+2}}d\kappa\right)du \tag{3.2.22}$$

hence

$$G_1(x,t) = \int_0^{+\infty}\left\{\int_0^t F_1^{(2\alpha)}\left[u(t-\tau)\right]F_2^{(\alpha)}\left[(2au)^{-1/\alpha}\tau\right]d\tau\right\}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-idux|\kappa|^{\beta+2}}d\kappa\right)du$$

$$+2a\int_0^{+\infty}\left\{\int_0^t F_1^{\alpha}\left[u(t-\tau)\right]F_2^{(2\alpha)}\left[(u^{-1/2\alpha})\tau\right]d\tau\right\}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-idux|\kappa|^{\beta+2}}d\kappa\right)du \tag{3.2.23}$$

By the same technique , we can obtain the expression of $G_2(x,t)$ in (3.2.6) as

$$G_2(\kappa,p) = \int_0^{+\infty} e^{-u(p^{2\alpha}+2ap^{\alpha}+d|\kappa|^{\beta+1})}du$$

$$= \int_0^{+\infty} e^{-up^{2\alpha}}e^{-2aup^{\alpha}}e^{-du|\kappa|^{\beta+1}}du$$

$$= \int_0^{+\infty} \ell\left\{F_2^{(2\alpha)}\left[(u^{-1/2\alpha})t\right]*F_2^{(\alpha)}\left[(2au)^{-1/\alpha}t\right]\right\}e^{-du|\kappa|^{\beta+1}}du$$

Now, we invert the Laplace and Fourier transform to obtain the relation

$$G_2(x,t) = \int_0^{+\infty}\left\{\int_0^t F_2^{(2\alpha)}\left[(u^{-1/2\alpha})(t-\tau)\right]F_2^{(\alpha)}\left[(2au)^{-1/\alpha}\tau\right]d\tau\right\}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-idux|\kappa|^{\beta+2}}d\kappa\right)du \tag{3.2.24}$$

So that, $u(x,t) = \int_{-\infty}^{+\infty} G_1(x-y,t)\phi(y)dy + \int_{-\infty}^{+\infty}\int_0^t G_2(x-y,t-\tau)f(y,\tau)d\tau dy$,

is the general solution of the time-fractional telegraph equation in a whole space domain

3.2.1 Special case :

If $\beta = 1$, in Eq.(3.1.1) with the initial and boundary condition (3.1.2) then we obtain

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} u(x,t) + 2a \frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = d \; {}_{-\infty}D_x^{\,2} u(x,t) + f(x,t), \quad t \in \Re^{+} \tag{3.2.25}$$

the solution of the Cauchy problem for the time-fractional telegraph equation (3.2.25) in a whole space domain can be found in [12, pp.3-6].

i.e. applying the Laplace transform with respect to t and Fourier transform with respect to x and using the initial and boundary conditions (3.1.2), we obtain the nonhomogeneous differential equation

$$u(\kappa, p) = \frac{p^{2\alpha-1} + 2ap^{\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + d\kappa^2}\phi(\kappa) + \frac{1}{p^{2\alpha} + 2ap^{\alpha} + d\kappa^2} f(\kappa, p)$$

$$= G_1(\kappa, p)\phi(\kappa) + G_2(\kappa, p) f(\kappa, p) \tag{3.2.26}$$

where

$$G_2(\kappa, p) = \frac{1}{p^{2\alpha} + 2ap^{\alpha} + d\kappa^2} \tag{3.2.27}$$

$$G_1(\kappa, p) = \frac{p^{2\alpha-1} + 2ap^{\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + d\kappa^2} = G_{1,1}(\kappa, p) + G_{1,2}(\kappa, p)$$

$$G_{1,1}(\kappa, p) = \frac{p^{2\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + d\kappa^2}, \quad G_{1,2}(\kappa, p) = \frac{2ap^{\alpha-1}}{p^{2\alpha} + 2ap^{\alpha} + d\kappa^2} \tag{3.2.28}$$

by the Fourier transform pair

$$e^{-c|x|} \overset{\Im}{\longleftrightarrow} \frac{2c}{c^2 + \kappa^2} \tag{3.2.29}$$

we have

$$G_{1,1}(x, p) = \frac{p^{2\alpha-1}}{2\sqrt{d(p^{2\alpha} + 2ap^{\alpha})}} e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}|x|} , \tag{3.2.30}$$

$$G_{1,2}(x, p) = \frac{2ap^{\alpha-1}}{2\sqrt{d(p^{2\alpha} + 2ap^{\alpha})}} e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}|x|} \tag{3.2.31}$$

and

$$G_2(x, p) = \frac{1}{2\sqrt{d(p^{2\alpha} + 2ap^{\alpha})}} e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}|x|} \tag{3.2.32}$$

Now, we invert the Fourier transform in Eq.(3.2.26) to obtain

$$u(x,t) = \int_{-\infty}^{+\infty} G_1(x-y,t)\phi(y)dy + \int_{-\infty}^{\infty}\int_{0}^{t} G_2(x-y,t-\tau) f(y,\tau) d\tau dy \tag{3.2.33}$$

where $G_1(x,t)$, $G_2(x,t)$ are the corresponding Green functions or fundamental solutions obtained when $\phi(x)=\delta(x),\ f(x)=0$ and $\phi(x)=0,\ f(x,t)=\delta(x)\delta(t)$ respectively.

3.3 The Solution of The "TFTE" in a Half-Space Domain

Similarly as in section (2.2) applying Fourier transform of the Weyl derivative (1.3.24) with respect to x , and Laplace transform of the Caputo derivative (1.3.15) with respect to t , in (3.1.1), respectively and the initial boundary conditions (3.1.3) , we obtain

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(\kappa,t)+2a\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(\kappa,t)=-d\left|\kappa\right|^{\beta+1}u(\kappa,t)+f(\kappa,t) \quad (3.3.1)$$

hence

$$p^{2\alpha}u(\kappa,p)+2ap^{\alpha}u(\kappa,p)=-d\left|\kappa\right|^{\beta+1}u(\kappa,p)+f(\kappa,p) \quad (3.3.2)$$

We can write $u(\kappa,p)$ explicitly as

$$u(\kappa,p)=\frac{1}{p^{2\alpha}+2ap^{\alpha}+d\left|\kappa\right|^{\beta+1}}f(\kappa,p) \quad (3.3.3)$$

By (3.2.6) we obtain

$$u(\kappa,p)=G_2(\kappa,p)f(\kappa,p) \quad (3.3.4)$$

Similarly, as in the previous section (2.2) making use of the formula given by (1.4.16) we can invert the Laplace transform in (3.2.6), to obtain Eq.(3.2.8), i.e

$$G_2(\kappa,t)=\frac{t^{\alpha-1}}{2\sqrt{(a^2-d\left|\kappa\right|^{\beta+1})}}\left[E_{\alpha,\alpha}(\lambda t^{\alpha})-E_{\alpha,\alpha}(\mu t^{\alpha})\right]$$

hence we can write Eq.(3.3.4) by Eq.(3.2.8) to obtain

$$u(\kappa,t)=G_2(\kappa,t)f(\kappa,p) \quad (3.3.5)$$

Now, we invert the Fourier transform in (3.3.5) to obtain

$$u(x,t)=\int_{-\infty}^{+\infty}\int_{0}^{t}G_2(x-y,t-\tau)f(y,\tau)d\tau dy \quad (3.3.6)$$

where $G_2(x,t)$ are the corresponding Green function or fundamental solution obtained when $f(x,t)=\delta(x)\delta(t)$ characterized by (3.2.24).

To more explain the previous paragraph :

when $f(x,t)=\delta(x)\delta(t)$ in the initial and boundary conditions (3.1.3) then we can write Eq.(3.3.5) as

$$u(\kappa,t)=G_2(\kappa,t)\delta(\kappa)\delta(p) \qquad (3.3.7)$$

but since as in (1.4.30), $\Im\{\delta(x),\kappa\}=1$ and (1.4.31), $\ell\{\delta(t);p\}=1$ then we can write Eq.(3.3.7) as

$$u(\kappa,t)=G_2(\kappa,t) \qquad (3.3.8)$$

Now, taking the inverse Fourier transform of Eq.(3.3.8) to obtain

$$u(x,t)=G_2(x,t)=\frac{t^{\alpha-1}}{2\sqrt{(a^2-d\,|x|^{\beta+1})}}\left[E_{\alpha,\alpha}(\lambda t^{\alpha})-E_{\alpha,\alpha}(\mu t^{\alpha})\right] \qquad (3.3.9)$$

3.3.1 Special case :

If $\beta=1$, in Eq.(3.1.1) with the initial and boundary conditions (3.1.3) then we obtain

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)+2a\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=d\;{}_{-\infty}D_x{}^{2}u(x,t)+f(x,t), \quad t\in\Re^{+} \qquad (3.3.10)$$

the solution of the Signaling problem for the time-fractional telegraph equation in a half space domain can be found in [12, pp. 6].

i.e. by application of the Laplace transform to Eq.(3.3.10) and initial and boundary condition (3.1.3) with $f(x,t)=0$, we obtain

$$\begin{aligned} {}_{-\infty}D_x{}^{2}u(x,p)&=\frac{p^{2\alpha}+2ap^{\alpha}}{d}u(x,p) \\ u(0,p)=g(p), &\qquad u(+\infty,p)=0 \end{aligned} \qquad (3.3.11)$$

with the solution

$$u(x,p)=g(p)e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}x}=\ell\{G_s(x,t)*g(t)\} \qquad (3.3.12)$$

where $G_s(x,t)$ is the Green function or the fundamental solution of the signaling problem obtained when $g(x)=\delta(x)$, which is characterized by

$$G_s(x,t)=e^{-\sqrt{((p^{2\alpha}+2ap^{\alpha})/d)}x} \qquad (3.3.13)$$

The inverse Laplace transform of Eq.(3.3.12) is obtained the solution of signaling problem

$$u(x,t)=G_s(x,t)*g(t)=\int_0^t G_s(x,t-\tau)g(\tau)d\tau \qquad (3.3.14)$$

from (3.2.30), (3.2.31) and (3.3.13), we recognize the relation

$$\frac{\partial}{\partial p}G_s(x,p) = -2\alpha x G_{1,1}(x,p) - \alpha x G_{1,2}(x,p), \quad x > 0 \tag{3.3.15}$$

Returning the space-time domain we obtain the relation

$$tG_s(x,t) = 2\alpha x G_{1,1}(x,t) + \alpha x G_{1,2}(x,t), \quad x,t > 0 \tag{3.3.16}$$

then we can obtain a representation for $G_s(x,t)$ and prove the negative prosperities .

3.4 The Solution of "TFTE" in a Bounded – Space Domain

By taking the finite Fourier sine transform(1.2.19) to (3.1.1) and applying the initial and boundary conditions (3.1.4) we obtain that

$$D_t^{2\alpha}\bar{u}(n,t) + 2aD_t^{\alpha}\bar{u}(n,t) = -\left|\frac{nd\pi}{L}\right|^{\beta+1}\bar{u}(n,t), \quad t > 0 \tag{3.4.1}$$

where $\bar{u}(n,t) = \int_0^L u(y,t)\sin\left(\frac{n\pi y}{L}\right)dy$, n is wave number (3.4.2)

and also applying the Laplace transform to (3.4.1) and using the initial conditions (3.1.4)

$$p^{2\alpha}\bar{u}(n,p) - p^{2\alpha-1}\bar{u}(n,0) - p^{2\alpha-2}\bar{u}_t(n,0) + 2ap^{\alpha}\bar{u}(n,p) - 2ap^{\alpha-1}\bar{u}(n,0) = \left|\frac{nd\pi}{L}\right|^{\beta+1}\bar{u}(n,p) \tag{3.4.3}$$

hence $$\bar{u}(n,p) = \frac{\left(p^{2\alpha-1} + 2ap^{\alpha-1}\right)\bar{u}(n,0)}{p^{2\alpha} + 2ap^{\alpha} + \left|\frac{nd\pi}{L}\right|^{\beta+1}} + \frac{p^{2\alpha-2}\bar{u}_t(n,0)}{p^{2\alpha} + 2ap^{\alpha} + \left|\frac{nd\pi}{L}\right|^{\beta+1}} \tag{3.4.4}$$

where $\bar{u}(n,0) = \int_0^L \phi(y)\sin\left(\frac{n\pi y}{L}\right)dy$ (3.4.5)

and $\bar{u}_t(n,0) = \int_0^L \varphi(y)\sin\left(\frac{n\pi y}{L}\right)dy$ (3.4.6)

We set $$\lambda_{\pm} = \frac{-2a \pm \sqrt{4a^2 - 4\left(nd\pi/L\right)^{\beta+1}}}{2} = -a \pm \sqrt{a^2 - \left(nd\pi/L\right)^{\beta+1}} \tag{3.4.7}$$

Then , $$p^{2\alpha} + 2ap^{\alpha} + \left(nd\pi/L\right)^{\beta+1} = (p^{\alpha} - \lambda_-)(p^{\alpha} - \lambda_+) \tag{3.4.8}$$

hence $$\frac{\left(p^{2\alpha-1} + 2ap^{\alpha-1}\right)}{p^{2\alpha} + 2ap^{\alpha} + \left(nd\pi/L\right)^{\beta+1}} = \frac{c_1 p^{\alpha-1}}{(p^{\alpha} - \lambda_-)} - \frac{c_2 p^{\alpha-1}}{(p^{\alpha} - \lambda_+)} \tag{3.4.9}$$

and $$\frac{p^{2\alpha-2}}{p^{2\alpha}+2ap^{\alpha}+\left(nd\pi/L\right)^{\beta+1}}=\frac{c_1 p^{\alpha-2}}{(p^{\alpha}-\lambda_+)}-\frac{c_2 p^{\alpha-2}}{(p^{\alpha}-\lambda_-)} \tag{3.4.10}$$

where $$c_1=\frac{\lambda_+}{\lambda_+-\lambda_-} \quad and \quad c_2=\frac{\lambda_-}{\lambda_+-\lambda_-} \tag{3.4.11}$$

Now, we can written $\bar{u}(n,p)$ as

$$\bar{u}(n,p)=\left(\frac{c_1 p^{\alpha-1}}{(p^{\alpha}-\lambda_-)}-\frac{c_2 p^{\alpha-1}}{(p^{\alpha}-\lambda_+)}\right)\bar{u}(n,0)+\left(\frac{c_1 p^{\alpha-2}}{(p^{\alpha}-\lambda_+)}-\frac{c_2 p^{\alpha-2}}{(p^{\alpha}-\lambda_-)}\right)\bar{u}_t(n,0) \tag{3.4.12}$$

to inverse the Laplace transform for (3.4.9) and (3.4.10) we use the formula defined in (1.4.14), i.e $\ell^{-1}\left\{\frac{p^{\alpha-\beta}}{p^{\alpha}\pm c};t\right\}=t^{\beta-1}E_{\alpha,\beta}\left(\mp ct^{\alpha}\right), \quad c>0$

Where $E_{\alpha,\beta}(z)$ is the so-called two parameter Mittag-Leffler function which is defined in (1.4.8),

we obtain

$$\ell^{-1}\left\{\frac{c_1 p^{\alpha-1}}{(p^{\alpha}-\lambda_-)}-\frac{c_2 p^{\alpha-1}}{(p^{\alpha}-\lambda_+)};t\right\}=\left[c_1 E_{\alpha}(\lambda_- t^{\alpha})-c_2 E_{\alpha}(\lambda_+ t^{\alpha})\right] \tag{3.4.13}$$

and

$$\ell^{-1}\left\{\frac{c_1 p^{\alpha-2}}{(p^{\alpha}-\lambda_+)}-\frac{c_2 p^{\alpha-2}}{(p^{\alpha}-\lambda_-)};t\right\}=t\left[c_1 E_{\alpha,2}(\lambda_+ t^{\alpha})-c_2 E_{\alpha,2}(\lambda_- t^{\alpha})\right] \tag{3.4.14}$$

So, we use inverse Laplace and finite Fourier sine transforms (3.4.13), (3.4.14) and (1.2.20) for (3.4.12) to obtain

$$\begin{aligned} u(x,t)&=\frac{2}{L}\sum_{n=1}^{\infty}\left[c_1 E_{\alpha}(\lambda_- t^{\alpha})-c_2 E_{\alpha}(\lambda_+ t^{\alpha})\right]\sin\left(\frac{n\pi x}{L}\right)\int_0^L \phi(y)\sin\left(\frac{n\pi y}{L}\right)dy \\ &+\frac{2}{L}\sum_{n=1}^{\infty}t\left[c_1 E_{\alpha,2}(\lambda_+ t^{\alpha})-c_2 E_{\alpha,2}(\lambda_- t^{\alpha})\right]\sin\left(\frac{n\pi x}{L}\right)\int_0^L \varphi(y)\sin\left(\frac{n\pi y}{L}\right)dy \end{aligned} \tag{3.4.15}$$

3.4.1 Special case :

If $\beta=1$, in Eq.(3.1.1) with the initial and boundary conditions (3.1.4) then we obtain

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)+2a\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=d\ {}_{-\infty}D_x^{\,2}u(x,t)+f(x,t), \quad t\in\Re^+ \tag{3.4.16}$$

the solution of the time-fractional telegraph equation in a bounded space domain can be found in [12, pp.7-8].

i.e. applying the finite Sine and Laplace transforms to Eq.(3.4.16) with $f(x,t)=0$ and initial and boundary conditions (3.1.4) we obtain

$$\bar{u}(n,p)=\frac{(p^{2\alpha-1}+2ap^{\alpha-1})\bar{u}(n,0)}{p^{2\alpha}+2ap^{\alpha}+(nd\pi/L)^2}+\frac{p^{2\alpha-2}\bar{u}_t(n,0)}{p^{2\alpha}+2ap^{\alpha}+(nd\pi/L)^2} \tag{3.4.17}$$

we set $\lambda_{\pm}=-a\pm\sqrt{a^2-(nd\pi/L)^2}$ then

$$p^{2\alpha}+2ap^{\alpha}+(nd\pi/L)^2=(p^{\alpha}-\lambda_-)(p^{\alpha}-\lambda_+) \tag{3.4.18}$$

to inverse the Laplace transform for (3.4.17) we use Eq.(1.4.14) to obtain

$$\ell^{-1}\left\{\frac{c_1p^{\alpha-1}}{(p^{\alpha}-\lambda_-)}-\frac{c_2p^{\alpha-1}}{(p^{\alpha}-\lambda_+)};t\right\}=\left[c_1E_{\alpha}(\lambda_-t^{\alpha})-c_2E_{\alpha}(\lambda_+t^{\alpha})\right] \tag{3.4.19}$$

and

$$\ell^{-1}\left\{\frac{c_1p^{\alpha-2}}{(p^{\alpha}-\lambda_+)}-\frac{c_2p^{\alpha-2}}{(p^{\alpha}-\lambda_-)};t\right\}=t\left[c_1E_{\alpha,2}(\lambda_+t^{\alpha})-c_2E_{\alpha,2}(\lambda_-t^{\alpha})\right] \tag{3.4.20}$$

where $c_1=\dfrac{\lambda_+}{\lambda_+-\lambda_-}$ *and* $c_2=\dfrac{\lambda_-}{\lambda_+-\lambda_-}$

So, we use inverse finite Fourier sine transforms for (3.4.17) to obtain

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left[c_1E_{\alpha}(\lambda_-t^{\alpha})-c_2E_{\alpha}(\lambda_+t^{\alpha})\right]\sin\left(\frac{n\pi x}{L}\right)\int_0^L\phi(y)\sin\left(\frac{n\pi y}{L}\right)dy$$

$$+\frac{2}{L}\sum_{n=1}^{\infty}t\left[c_1E_{\alpha,2}(\lambda_+t^{\alpha})-c_2E_{\alpha,2}(\lambda_-t^{\alpha})\right]\sin\left(\frac{n\pi x}{L}\right)\int_0^L\varphi(y)\sin\left(\frac{n\pi y}{L}\right)dy \tag{3.4.21}$$

CHAPTER 4

Fractional Green Function For The Fractional Evolution Equation

4.1 Introduction

Fractional order partial differential equations, as generalization of classical integer order partial differential equations, are increasingly used to model problems in fluid mechanics, viscoelasticity, chemistry, physics, finance and other areas of application. Consequently, considerable attention has been given to the solutions of fractional ordinary and partial differential equations of physical interest [11,21]. Several authors including Poldlubny [39], Beyer and Kempfle [1], Schneider and Wyss [48], Mainardi [28] and Huang and Liu [13] discussed some examples of homogeneous fractional ordinary differential equations, homogeneous fractional diffusion and wave equations. The main focus of the above authors is on the homogeneous fractional differential equations. Recently, Debnath and Bhatta [6] solve some linear inhomogeneous fractional partial differential equations in fluid mechanics. These equations include diffusion and wave equation, Burgers equation, Kortweg and de Vries (KdV) equation, KdVBurgers equation, Klein-Gorden equation, telegraph equation and Stokes-Ekman equation. The solutions are obtained with help of the joint Laplace and Fourier transform combined with the Mittag-Leffler function. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, the Fourier transform method, the iteration method [45] and the operational method [21]. Furthermore, one of the most useful methods is the Green function method. Schneider and Wyss [48] obtained the Green function for the time fractional diffusion equation. Mainardi et al. [28] obtained the Green function for the space-time fractional diffusion equation. Huang and Liu [13] obtained the Green function for the fractional advection-dispersion equation. However, most of these authors derived the Green function in terms of Fox functions .

In this chapter, we derive the fractional Green function for the time-fractional evolution equation that generalizes many partial differential equation in fluid mechanics. The main objective of the present chapter is to use the Green function method to obtain solutions of the time-fractional Wave equation, linearized time-fractional Burgers equation, linear time-fractional KdV equation.

4.2 Fractional Evolution Equation

Let us consider the general linear inhomogeneous fractional evolution equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)+\sum_{k=0}^{n-1}a_k\ {}_{-\infty}D_x^{\ \beta+k}u(x,t)=q(x,t), \quad t>0, \quad x\in\Re \tag{4.2.1}$$

where, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is fractional derivative of the Caputo sense of order α, ${}_{-\infty}D_x^{\beta+k}$ is Weyl fractional derivative and $0<\alpha\le 1$, $0<\beta\le 1$

$a_k\ (k=0,1,\dots,n-1),\ n\in\mathbb{N}$ are constants , subject to the initial and boundary conditions

$$u(x,0)=f(x)$$
$$u(x,t)\to 0 \quad as \quad |x|\to\infty \tag{4.2.2}$$

The Laplace transform of (4.2.1)with respect to t , subject to (4.2.2) is

$$p^{\alpha}u(x,p)-p^{\alpha-1}f(x)+a_0\,{}_{-\infty}D_x^{\beta}u(x,p)+.....+a_n\,{}_{-\infty}D_x^{\beta+n-1}u(x,p)=q(x,p) \tag{4.2.3}$$

Furthermore the Fourier transform of (4.2.3) with respect to x is

$$p^{\alpha}u(\kappa,p)-p^{\alpha-1}f(\kappa)+a_0(i\kappa)^{\beta}u(\kappa,p)+.....+a_n(i\kappa)^{\beta+n-1}u(\kappa,p)=q(\kappa,p) \tag{4.2.4}$$

which implies

$$u(\kappa,p)=\frac{p^{\alpha-1}f(\kappa)}{p^{\alpha}+a^2}+\frac{q(\kappa,p)}{p^{\alpha}+a^2} \tag{4.2.5}$$

where $$a^2=\sum_{j=0}^{n-1}(i\kappa)^{j+\beta}a_j \quad , \quad i=\sqrt{-1} \tag{4.2.6}$$

The fractional Green function $G_{\alpha}(x,t)$ of the fractional evolution equation (4.2.1) can be found by the Laplace and Fourier inversion of the expression

$$G_{\alpha}(\kappa,p)=\frac{1}{p^{\alpha}+a^2} \tag{4.2.7}$$

also the Green function $G(x,t)$ of the Evolution equation

$$\frac{\partial}{\partial t}u(x,t)+\sum_{k=0}^{n-1}a_k\,{}_{-\infty}D_x^{\beta+k}u(x,t)=q(x,t), \quad t>0, \quad x\in\Re \tag{4.2.8}$$

can be found by the Laplace and Fourier inversion of the expression

$$G(\kappa,p)=\frac{1}{p+a^2} \tag{4.2.9}$$

in view of (4.2.7) and (4.2.9) , the fractional Green function $G_{\alpha}(x,t)$ and Green function $G(x,t)$ are related by

$$G_\alpha(x,p) = G(x,p^\alpha) \tag{4.2.10}$$

by using the relation (1.3.20) between Mellin and Laplace transform, we obtain

$$\overset{*}{G}_\alpha(x,s) = \frac{1}{\Gamma(1-s)}\int_0^\infty p^{-s} G_\alpha(x,p)dp \tag{4.2.11}$$

$$= \frac{1}{\Gamma(1-s)}\int_0^\infty p^{-s} G(x,p^\alpha)dp \tag{4.2.12}$$

Let , $p^\alpha = p_0$ thus $\Rightarrow \alpha p^{\alpha-1}dp = dp_0$

$p: 0 \to \infty$ thus $p_0 : 0 \to \infty$

hence $$\overset{*}{G}_\alpha(x,s) = \frac{1}{\Gamma(1-s)}\int_0^\infty p_0^{\frac{-s}{\alpha}} G(x,p_0) p_0^{\frac{-(\alpha-1)}{\alpha}} \frac{dp_0}{\alpha} \tag{4.2.13}$$

$$= \frac{1}{\alpha}\frac{1}{\Gamma(1-s)}\int_0^\infty p_0^{-\left(\frac{s+\alpha-1}{\alpha}\right)} G(x,p_0)dp_0 \tag{4.2.14}$$

$$\overset{*}{G}_\alpha(x,s) = \frac{1}{\alpha}\frac{\Gamma(1-\frac{s+\alpha-1}{\alpha})}{\Gamma(1-s)}\overset{*}{G}(x,\frac{s+\alpha-1}{\alpha}) \tag{4.2.15}$$

Define $\Phi(t) = W(-\alpha,0;-t)$ (4.2.16)

where $W(-\alpha,0;-t)$ is the Wright function defined in (1.4.17)

by using the Mellin transform for the Wright function $W(-\alpha,0;-t)$ (1.4.18), i.e $\int_0^\infty t^{s-1} W(-\alpha,0;-t)dt = \frac{\Gamma(s)}{\Gamma(\alpha s)}$ to obtain

$$\overset{*}{\Phi}(s) = \frac{\Gamma(s)}{\Gamma(\alpha s)} \tag{4.2.17}$$

in view of (4.2.17) and (4.2.15) we can written

$$\overset{*}{G}_\alpha(x,s) = \frac{1}{\alpha}\overset{*}{\Phi}(1-\frac{s+\alpha-1}{\alpha})\overset{*}{G}(x,\frac{s+\alpha-1}{\alpha}) \tag{4.2.18}$$

$$= \frac{1}{\alpha}\overset{*}{\Phi}(\frac{1-s}{\alpha})\overset{*}{G}(x,1-\left(\frac{1-s}{\alpha}\right))$$

$$= \frac{1}{\alpha}\overset{*}{\Phi}(\frac{s-1}{-\alpha})\overset{*}{G}(x,1-\left(\frac{s-1}{-\alpha}\right)) \tag{4.2.19}$$

by (1.2.13) , (1.2.14) and (1.2.16) of Mellin transform

i.e : $M\left\{t^{-\alpha}f\left(t\right);s\right\}=M\left\{f(t);s-\alpha\right\}=f^{*}\left(s-\alpha\right)=\int_{0}^{\infty}f(t)t^{s-\alpha-1}dt$,

$$M\left\{f\left(t^{\gamma}\right);s\right\}=\frac{1}{\gamma}f^{*}\left(\frac{s}{\gamma}\right)=\frac{1}{\gamma}\int_{0}^{\infty}f(t)t^{\frac{s}{\gamma}-1}$$

and $M\left\{\int_{0}^{\infty}f\left(t\tau\right)g\left(\tau\right)d\tau;s\right\}=f^{*}\left(s\right)g^{*}\left(1-s\right)$

we can obtain

$$\overset{*}{G}_{\alpha}(x,s)=M\left(t^{-1}\int_{0}^{\infty}G(x,z)\Phi(t^{-\alpha}z)dz\right) \tag{4.2.20}$$

The inverse of Mellin transform for (4.2.20) gives

$$G_{\alpha}(x,s)=t^{-1}\int_{0}^{\infty}G(x,z)\Phi(t^{-\alpha}z)dz \tag{4.2.21}$$

Note that, the solution of the fractional evolution equation (4.2.1), subject to the initial and boundary condition (4.2.2) is given by

$$u(x,t)=\frac{1}{\sqrt{2\pi}\Gamma(1-\alpha)}\int_{-\infty}^{\infty}f(x-\xi)\int_{0}^{t}G_{\alpha}(\xi,t)(t-\tau)^{-\alpha}d\tau d\xi$$
$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\kappa x}\int_{0}^{t}G_{\alpha}(\kappa,\tau)q(\kappa,t-\tau)d\tau d\kappa \tag{4.2.22}$$

4.2.1 Special case

If $\beta=1$ in Eq.(4.2.1) with the initial and boundary conditions (4.2.2) then we obtain

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)+\sum_{k=0}^{n-1}a_{k}\ {}_{-\infty}D_{x}^{\ k+1}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re,\quad 0<\alpha<1 \tag{4.2.23}$$

the general solution of the fractional evolution equation (4.2.23) obtained by Momani and Odibat [35, pp.170-172] .

i.e. applying Laplace transform with respect to t and Fourier transform with respect to x to Eq.(4.2.23) with the initial and boundary conditions (4.2.2) we obtain

$$u(\kappa,p)=\frac{p^{\alpha-1}f(\kappa)}{p^{\alpha}+a^{2}}+\frac{q(\kappa,p)}{p^{\alpha}+a^{2}} \tag{4.2.24}$$

where $a^{2}=\sum_{j=0}^{n-1}(i\kappa)^{j}a_{j}\quad,\quad i=\sqrt{-1}$ (4.2.25)

The fractional Green function $G_\alpha(x,t)$ of Eq.(4.2.23) can be found by the Laplace and Fourier inversion of the expression

$$G_\alpha(\kappa, p) = \frac{1}{p^\alpha + a^2} \qquad (4.2.26)$$

The Green function $G(x,t)$ of the evolution equation

$$\frac{\partial}{\partial t} u(x,t) + \sum_{k=0}^{n-1} a_k \; {}_{-\infty}D_x^{\,k+1} u(x,t) = q(x,t), \quad t > 0, \quad x \in \Re, \qquad (4.2.27)$$

can be found by the Laplace and Fourier inversion of the expression

$$G(\kappa, p) = \frac{1}{p + a^2} \qquad (4.2.28)$$

The fractional Green function $G_\alpha(x,t)$ of the time-fractional evolution equation (4.2.23) related to the Green function $G(x,t)$ of the evolution equation (4.2.27) can be found as

$$G_\alpha(x,s) = t^{-1} \int_0^\infty G(x,z)\Phi(t^{-\alpha} z)\,dz \qquad (4.2.29)$$

The solution of the time-fractional evolution equation (4.2.23), subject to the initial and boundary condition (4.2.2) is given by

$$u(x,t) = \frac{1}{\sqrt{2\pi}\,\Gamma(1-\alpha)} \int_{-\infty}^{\infty} f(x-\xi) \int_0^t G_\alpha(\xi,t)(t-\tau)^{-\alpha}\, d\tau\, d\xi$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\kappa x} \int_0^t G_\alpha(\kappa,\tau) q(\kappa, t-\tau)\, d\tau\, d\kappa \qquad (4.2.30)$$

Now, we use the fractional Green function that obtained in the relation (4.2.21) to derive the solutions of the inhomogeneous fractional Wave equation, inhomogeneous linear fractional Burgers equation, and also inhomogeneous linear fractional KdV equation .

4.3 Fractional Wave Equation

Consider the one dimensional linear inhomogeneous fractional wave equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) - c^2 \; {}_{-\infty}D_x^{\,\beta+1} u(x,t) = q(x,t), \quad t > 0, \quad x \in \Re \qquad (4.3.1)$$

where c is constant, $0 < \alpha \le 1$ and $0 < \beta \le 1$

also $q(x,t)$ is a source term, subject to initial and boundary conditions

$$u(x,0) = f(x)$$
$$u(x,t) \to 0 \quad as \quad |x| \to \infty \tag{4.3.2}$$

Now, by taking the Laplace and Fourier transforms for the classical wave equation

$$\frac{\partial}{\partial t}u(x,t) - c^2 {}_{-\infty}D_x^{\beta+1}u(x,t) = q(x,t), \quad t > 0, \quad x \in \Re \tag{4.3.3}$$

we get

$$pu(\kappa,p) - c^2(i\kappa)^{\beta+1}u(\kappa,p) = q(\kappa,p) \tag{4.3.4}$$

which implies

$$u(\kappa,p) = \frac{q(\kappa,p)}{p - c^2(i\kappa)^{\beta+1}} \tag{4.3.5}$$

The Laplace and Fourier transforms of the Green function for the classical wave equation takes the form

$$G(\kappa,p) = \frac{1}{p - c^2(i\kappa)^{\beta+1}} \tag{4.3.6}$$

The inverse Laplace transform of some useful formulas produces Mittag-Leffler function of one parameter[39], such as

$$\ell^{-1}\left\{\frac{p^{\alpha-1}}{p^{\alpha}+c};t\right\} = E_{\alpha}(-ct^{\alpha}) \tag{4.3.7}$$

where $E_{\alpha}(z)$ is the Mittag-Leffler function defined in (1.4.7)

hence, we can get

$$G(\kappa,t) = E_1((i\kappa)^{\beta+1}c^2t) = e^{(i\kappa)^{\beta+1}c^2t} \tag{4.3.8}$$

where $E_1(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z$ and $G(\kappa,t)$ is the Fourier transform of the Green function for the classical wave equation (4.3.3) .

according to relation (4.2.21), the Fourier transform of the fractional Green function for the fractional wave equation (4.3.1) can be found as

$$G_{\alpha}(\kappa,t)=t^{-1}\int_{0}^{\infty}G(\kappa,z)\Phi(t^{-\alpha}z)dz$$

$$=t^{-1}\int_{0}^{\infty}e^{(i\kappa)^{\beta+1}c^{2}t}\Phi(t^{-\alpha}z)dz \qquad (4.3.9)$$

hence, the solution of the inhomogeneous fractional wave equation (4.3.1) is given by the inverse Fourier transform of the following expression

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_{0}^{t}G_{\alpha}(\kappa,\tau)(t-\tau)^{-\alpha}d\tau+\int_{0}^{t}G_{\alpha}(\kappa,\tau)q(\kappa,t-\tau)d\tau \qquad (4.3.10)$$

That is

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{1}{\tau(t-\tau)^{\alpha}}\int_{0}^{\infty}e^{(i\kappa)^{\beta+1}c^{2}t}\Phi(t^{-\alpha}z)dz\,d\tau+\int_{0}^{t}\frac{q(\kappa,t-\tau)}{\tau}\int_{0}^{\infty}e^{(i\kappa)^{\beta+1}c^{2}t}\Phi(t^{-\alpha}z)dz\,d\tau$$

(4.3.11)

Therefore, the solution of the inhomogeneous fractional wave equation (4.3.1) is given by

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\kappa x}\left[\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{1}{\tau(t-\tau)^{\alpha}}\int_{0}^{\infty}e^{(i\kappa)^{\beta+1}c^{2}t}\Phi(t^{-\alpha}z)dz\,d\tau\right.$$

$$\left.+\int_{0}^{t}\frac{q(\kappa,t-\tau)}{\tau}\int_{0}^{\infty}e^{(i\kappa)^{\beta+1}c^{2}t}\Phi(t^{-\alpha}z)dz\,d\tau\right]d\kappa \qquad (4.3.12)$$

4.3.1 Special case

If $\beta=1$ in Eq.(4.3.1), with the initial and boundary conditions (4.3.2) we obtain

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)-c^{2}\,{}_{-\infty}D_{x}^{2}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \qquad (4.3.13)$$

The solution of the fractional wave equation (4.3.13) obtained by Momani and Odibat [35, pp.172-174] .

i.e. the Green function of the classical wave equation

$$\frac{\partial}{\partial t}u(x,t)-c^{2}\,{}_{-\infty}D_{x}^{2}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \qquad (4.3.14)$$

takes the form

$$G(x,t)=\frac{1}{\sqrt{2c^{2}t}}e^{\frac{-x^{2}}{4c^{2}t}} \qquad (4.3.15)$$

The fractional Green function of the fractional wave equation (4.3.13) can be found as

$$G_\alpha(x,s)=t^{-1}\int_0^\infty G(x,z)\Phi(t^{-\alpha}z)dz$$

$$=\frac{-\sqrt{\pi}x}{\sqrt{2}c^2t}W_{(-\alpha,0),(2,2)}\left(\frac{t^{-\alpha}x^2}{c^2}\right) \qquad (4.3.16)$$

The solution of the fractional wave equation (4.3.13) is given by

$$u(x,t)=\frac{-1}{2\Gamma(1-\alpha)}\int_{-\infty}^{\infty} f(x-\xi)\int_0^t \frac{\xi}{c^2\tau}W_{(-\alpha,0),(2,2)}\left(\frac{\tau^{-\alpha}\xi^2}{c^2}\right)(t-\tau)^{-\alpha}d\tau d\xi$$

$$-\frac{1}{2}\int_{-\infty}^{\infty} e^{i\kappa x}\int_0^t \frac{k}{c^2\tau}W_{(-\alpha,0),(2,2)}\left(\frac{\tau^{-\alpha}k^2}{c^2}\right)q(\kappa,t-\tau)d\tau d\kappa \qquad (4.3.17)$$

In case of $q(x,t)=0$ in Eq.(4.3.13) then the solution (4.3.17) takes the form

$$u(x,t)=\frac{-t^{-\alpha}}{2c^2}\int_{-\infty}^{\infty} f(x-\xi)\sum_{k=0}^{\infty}\frac{t^{-\alpha k}\xi^{2k+1}\Gamma(\alpha(k+1))}{(2k+1)!c^{2k}}\frac{\sin(\alpha\pi(k+1))}{\pi}d\xi \qquad (4.3.18)$$

4.4 Fractional Burgers Equation

Consider the one-dimensional linear inhomogeneous fractional Burgers equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)+c\,{}_{-\infty}D_x{}^{\beta}u(x,t)-v\,{}_{-\infty}D_x{}^{\beta+1}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \qquad (4.4.1)$$

where c is constant , $0<\alpha\le 1$, $0<\beta\le 1$, $v>0$ is the kinematic viscosity and $q(x,t)$ is a source term this equation is considered subject to initial and boundary conditions

$$u(x,0)=f(x)$$

$$u(x,t)\to 0 \quad as \quad |x|\to\infty \qquad (4.4.2)$$

Now, by taking the Laplace and Fourier transform for the classical burgers equation

$$\frac{\partial}{\partial t}u(x,t)+c\,{}_{-\infty}D_x{}^{\beta}u(x,t)-v\,{}_{-\infty}D_x{}^{\beta+1}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \qquad (4.4.3)$$

we get

$$pu(\kappa,p)+c(i\kappa)^\beta u(\kappa,p)-v(i\kappa)^{\beta+1}u(\kappa,p)=q(\kappa,p) \qquad (4.4.4)$$

which implies

$$u(\kappa,p)=\frac{q(\kappa,p)}{p+c(i\kappa)^{\beta}-v(i\kappa)^{\beta+1}} \tag{4.4.5}$$

The Laplace and Fourier transforms of the Green function for the classical burgers equation takes the form

$$G(\kappa,p)=\frac{1}{p+c(i\kappa)^{\beta}-v(i\kappa)^{\beta+1}} \tag{4.4.6}$$

by using the inverse Laplace transform (4.3.7) we get

$$G(\kappa,t)=E_1\left(-(c(i\kappa)^{\beta}-v(i\kappa)^{\beta+1})t\right)=e^{(i\kappa)^{\beta}(v(i\kappa)-c)t} \tag{4.4.7}$$

which is the Fourier transform of the Green function for the classical burgers equation (4.4.3) .

According to relation (4.2.21), the Fourier transform of the fractional Green function for the fractional burgers equation (4.4.1) can be found as

$$G_{\alpha}(\kappa,t)=t^{-1}\int_0^{\infty}G(\kappa,z)\Phi(t^{-\alpha}z)dz$$

$$=t^{-1}\int_0^{\infty}e^{(i\kappa)^{\beta}(v(i\kappa)-c)t}\Phi(t^{-\alpha}z)dz \tag{4.4.8}$$

hence, the solution of the inhomogeneous fractional burgers equation (4.4.1) is given by the inverse Fourier transform of the following expression

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_0^{t}G_{\alpha}(\kappa,\tau)(t-\tau)^{-\alpha}d\tau+\int_0^{t}G_{\alpha}(\kappa,\tau)q(\kappa,t-\tau)d\tau \tag{4.4.9}$$

That is

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_0^{t}\frac{1}{\tau(t-\tau)^{\alpha}}\int_0^{\infty}e^{(i\kappa)^{\beta}(v(i\kappa)-c)t}\Phi(t^{-\alpha}z)dz\,d\tau$$

$$+\int_0^{t}\frac{q(\kappa,t-\tau)}{\tau}\int_0^{\infty}e^{(i\kappa)^{\beta}(v(i\kappa)-c)t}\Phi(t^{-\alpha}z)dz\,d\tau \tag{4.4.10}$$

Therefore, the solution of the inhomogeneous fractional Burgers equation (4.4.1) given by

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\kappa x}\left[\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_0^{t}\frac{1}{\tau(t-\tau)^{\alpha}}\int_0^{\infty}e^{(i\kappa)^{\beta}(v(i\kappa)-c)t}\Phi(t^{-\alpha}z)dz\,d\tau\right.$$

$$+\int_0^t \frac{q(\kappa,t-\tau)}{\tau}\int_0^\infty e^{(i\kappa)^\beta(v(i\kappa)-c)t}\Phi(t^{-\alpha}z)dz\,d\tau\Bigg]d\kappa \tag{4.4.11}$$

4.4.1 Special case

If $\beta=1$ in Eq.(4.4.1), with the initial and boundary conditions (4.4.2) we obtain

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)+c\ {}_{-\infty}D_x^{\ 1}u(x,t)-v\ {}_{-\infty}D_x^{\ 2}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re,\quad 0<\alpha<1 \tag{4.4.12}$$

The solution of the fractional burgers equation (4.4.12) obtained by Momani and Odibat [35, pp.174-176] .

i.e. the Fourier transform of the Green function for the classical burgers equation

$$\frac{\partial}{\partial t}u(x,t)+c\ {}_{-\infty}D_x^{\ 1}u(x,t)-v\ {}_{-\infty}D_x^{\ 2}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \tag{4.4.13}$$

can be written as

$$G(\kappa,t)=e^{-ickt}G^*(\kappa,t) \tag{4.4.14}$$

where

$$G^*(x,t)=\frac{1}{\sqrt{2vt}}e^{\frac{-x^2}{4vt}} \tag{4.4.15}$$

The fractional Green function $G_\alpha(x,t)$ of the time-fractional burgers equation (4.4.12) takes the form

$$G_\alpha(x,t)=\frac{-\sqrt{\pi}(x-ct)}{\sqrt{2}vt}W_{(-\alpha,0),(2,2)}\left(\frac{t^{-\alpha}(x-ct)^2}{v}\right) \tag{4.4.16}$$

The solution of the time-fractional burgers equation (4.4.12) is given by

$$u(x,t)=\frac{-1}{2\Gamma(1-\alpha)}\int_{-\infty}^{\infty}f(x-\xi)\int_0^t\frac{\xi-ct}{v\tau}W_{(-\alpha,0),(2,2)}\left(\frac{\tau^{-\alpha}(\xi-ct)^2}{v}\right)(t-\tau)^{-\alpha}d\tau d\xi$$

$$-\frac{1}{2}\int_{-\infty}^{\infty}e^{i\kappa x}\int_0^t\frac{k-ct}{v\tau}W_{(-\alpha,0),(2,2)}\left(\frac{\tau^{-\alpha}(k-ct)^2}{v}\right)q(\kappa,t-\tau)d\tau d\kappa \tag{4.4.17}$$

In case of $q(x,t)=0$ in Eq.(4.4.12) the solution (4.4.17) takes the form

$$u(x,t)=\frac{-1}{2}\int_{-\infty}^{\infty}f(x-\xi)\sum_{k=0}^{\infty}\frac{1}{\Gamma(-\alpha k)(2k+1)!v^{k+1}}\sum_{n=0}^{2k+1}\binom{2k+1}{n}\frac{(-1)^n c^n\Gamma(n-\alpha k)}{\Gamma(n-\alpha k-\alpha+1)}\xi^{2k+1}\tau^{n-\alpha k-\alpha}d\xi \tag{4.4.18}$$

4.5 Fractional Kortweg and de Vries (KdV) Equation

Consider the one-dimensional linear inhomogeneous fractional KdV equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)+c\,{}_{-\infty}D_x{}^{\beta}u(x,t)+b\,{}_{-\infty}D_x{}^{\beta+2}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \tag{4.5.1}$$

where c and b are constant , $0<\alpha\leq 1$, $0<\beta\leq 1$ and $q(x,t)$ is a source term, subject to initial and boundary conditions

$$\begin{aligned} &u(x,0)=f(x) \\ &u(x,t)\to 0 \quad as \quad |x|\to\infty \end{aligned} \tag{4.5.2}$$

Now, by taking the Laplace and Fourier transform for the classical KdV equation

$$\frac{\partial}{\partial t}u(x,t)+c\,{}_{-\infty}D_x{}^{\beta}u(x,t)+b\,{}_{-\infty}D_x{}^{\beta+2}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \tag{4.5.3}$$

we get

$$pu(\kappa,p)+c(i\kappa)^{\beta}u(\kappa,p)+b(i\kappa)^{\beta+2}u(\kappa,p)=q(\kappa,p) \tag{4.5.4}$$

which implies

$$u(\kappa,p)=\frac{q(\kappa,p)}{p+c(i\kappa)^{\beta}+b(i\kappa)^{\beta+2}} \tag{4.5.5}$$

The Laplace and Fourier transforms of the Green function for the classical KdV equation takes the form

$$G(\kappa,p)=\frac{1}{p+c(i\kappa)^{\beta}+B(i\kappa)^{\beta+2}} \tag{4.5.6}$$

using the inverse Laplace transform (4.3.7) we get

$$G(\kappa,t)=E_1\left(-(c(i\kappa)^{\beta}+b(i\kappa)^{\beta+2})t\right)=e^{(i\kappa)^{\beta}(b\kappa^2-c)t} \tag{4.5.7}$$

which is the Fourier transform of the Green function for the classical KdV equation (4.5.3)

According to relation (4.2.21), the Fourier transform of the fractional Green function for the fractional KdV equation (4.5.1) can be found as

$$G_{\alpha}(\kappa,t)=t^{-1}\int_0^{\infty}G(\kappa,z)\Phi(t^{-\alpha}z)dz$$

$$=t^{-1}\int_0^\infty e^{(i\kappa)^\beta (b\kappa^2-c)t}\,\Phi(t^{-\alpha}z)\,dz \qquad (4.5.8)$$

hence, the solution of the inhomogeneous fractional KdV equation (4.5.1) is given by the inverse Fourier transform of the following expression

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_0^t G_\alpha(\kappa,\tau)(t-\tau)^{-\alpha}\,d\tau+\int_0^t G_\alpha(\kappa,\tau)q(\kappa,t-\tau)\,d\tau \qquad (4.5.9)$$

That is

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_0^t \frac{1}{\tau(t-\tau)^\alpha}\int_0^\infty e^{(i\kappa)^\beta (b\kappa^2-c)t}\,\Phi(t^{-\alpha}z)\,dz\,d\tau$$

$$+\int_0^t \frac{q(\kappa,t-\tau)}{\tau}\int_0^\infty e^{(i\kappa)^\beta (b\kappa^2-c)t}\,\Phi(t^{-\alpha}z)\,dz\,d\tau \qquad (4.5.10)$$

Therefore, the solution of the inhomogeneous fractional KdV equation (4.5.1) given by

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\kappa x}\left[\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_0^t \frac{1}{\tau(t-\tau)^\alpha}\int_0^\infty e^{(i\kappa)^\beta (b\kappa^2-c)t}\,\Phi(t^{-\alpha}z)\,dz\,d\tau\right.$$

$$\left.+\int_0^t \frac{q(\kappa,t-\tau)}{\tau}\int_0^\infty e^{(i\kappa)^\beta (b\kappa^2-c)t}\,\Phi(t^{-\alpha}z)\,dz\,d\tau\right]d\kappa \qquad (4.5.11)$$

4.5.1 Special case

If $\beta=1$ in Eq.(4.5.1), with the initial and boundary conditions (4.5.2) we obtain

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)+c\,{}_{-\infty}D_x^{\,1}u(x,t)+b\,{}_{-\infty}D_x^{\,3}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \qquad (4.5.12)$$

the solution of the fractional Kortweg and de Vries (KdV) equation (4.5.12) obtained by Momani and Odibat [35, pp.176-177] .

i.e. the Fourier transform of the Green function for the classical KdV equation

$$\frac{\partial}{\partial t}u(x,t)+c\,{}_{-\infty}D_x^{\,1}u(x,t)+b\,{}_{-\infty}D_x^{\,3}u(x,t)=q(x,t),\quad t>0,\quad x\in\Re \qquad (4.5.13)$$

can be written as

$$G(\kappa,t)=e^{i(bk^3-ck)t} \qquad (4.5.14)$$

the Fourier transform of the fractional Green function for the time-fractional KdV equation (4.5.12) can be found as

$$G_{\alpha}(\kappa,t)=t^{-1}\int_{0}^{\infty}e^{i(bk^{3}-ck)t}\Phi(t^{-\alpha}z)dz \tag{4.5.15}$$

The solution of the time-fractional KdV equation (4.5.12) is given by

$$u(\kappa,t)=\frac{f(\kappa)}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{1}{\tau(t-\tau)^{\alpha}}\int_{0}^{\infty}e^{i(bk^{3}-ck)t}\Phi(t^{-\alpha}z)dz\,d\tau$$

$$+\int_{0}^{t}\frac{q(\kappa,t-\tau)}{\tau}\int_{0}^{\infty}e^{i(bk^{3}-ck)t}\Phi(t^{-\alpha}z)dz\,d\tau \tag{4.5.16}$$

CHAPTER 5

Conclusion and Recommendations

The time-fractional diffusion equation is obtained from the standard diffusion equation by replacing the first order time derivative by a fractional derivative of order $0 < \alpha \leq 1$. using the Laplace and Fourier transforms, we derive the complete solution of this time fractional diffusion equation. Also the fractional diffusion equation is obtained from the time-fractional diffusion equation by replacing the second order space derivative by Weyl fractional derivative of order $\beta + 1$, $0 < \beta \leq 1$. using the Laplace and Fourier transforms, we derive the complete solution of this fractional diffusion equation.

In chapter 3, we presented the solution of the time-fractional telegraph equation "TFTE". The fundamental solution for "TFTE" in a whole-space domain and "TFTE" in a half-space domain obtained by using Fourier-Laplace transforms and there inverse transforms getting the Green function which help us to reach to the solution. on the other hand, the solution in the form of a series for the "TFTE" in abounded-space domain is derived by the Sine-Laplace transforms methods.

Furthermore, fractional evolution equation is obtained from the time-fractional evolution equation by replacing the n^{th} order space derivative by Weyl fractional derivative of order $\beta + n - 1$, $0 < \beta \leq 1$. using integral transforms Laplace, Fourier and mellin and there inverse transforms to obtain the fractional Green function of the fractional evolution equation which help us reach to the fundamental solution of the fractional evolution equation. using the relationship between fractional Green function of the fractional evolution equation and the Green functions of the classical wave equation, classical burgers equation and classical KdV equation to obtained the fundamental solutions of the fractional wave equation, fractional burgers equation and fractional KdV equation.

For future work, I can recommend the following

(i) applying Caputo fractional derivative for the time and Weyl fractional derivative for the space to various fractional differential equations.

(ii) Using the methods of Laplace transform, Fourier transform and mellin transform to achieve the explicit solutions.

(iii) For researchers in numerical analysis, I recommend to solve numerically the integral formulas of the solutions obtained.

(iv) Try to solve the fractional differential equations solved in this research for distributed order and get the final solution in which our solutions could be special cases and get more related results.

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