Declaration

The title and the contents of this Thesis is a generalization of some existing problems, hence, we declare that all results are new and belongs to the authors of the Thesis.

Fayez A. Albhaisi

Dedication

To my parents, sisters, brothers, friends and Colleagues with love and respect.

Fayez A. Albhaisi

Acknowledgments

After thanking Allah, who granted me the ability to finish this work, I am indebt to my supervisor Dr. Mohammed Matar (Al-Azhar University-Gaza), for guiding me during the last months of work in this master Thesis and for his valuable suggestions. Finally, with all my heart, I would also like to thank again my parents for patience, support and love in every moment throughout my education. Thank you all very much indeed.

Fayez A. Albhaisi

Abstract

The existence of solutions for nonlinear fractional differential equations and inclusions of order $q \in (4,5]$ with anti-periodic and integral boundary conditions are studied as a generalization of many previous researches. In the case of inclusion, the existence results are established for convex as well as nonconvex multivalued maps. Our results are based on some fixed point theorems such as Banach, Kranselliski, Leray-Schauder degree theory, nonlinear alternative of Leray-Schauder type, and Covitz and Nadler. The theorems are illustrated by some examples.

Keywords: Existence, Fractional differential equations, Antiperiodic boundary condition, Multivalued map.

ملخص الرسالة

هذه الرسالة تدرس وجود حلول للمعادلات التفاضلية الكسرية الغير خطية برتبة الفا خلال الفترة (٤،٥] مع شروط حدودية تكاملية و غير دورية. في حالة المعادلات الكسرية المتضمنة، أسست النتائج للدوال متعددة القيم والمحدبة من ناحية و الغير محدبة من ناحية اخرى. نتائجنا بنيت على أساس بعض نظريات النقطة الثابتة مثل نظرية بناخ، وكرانسيلوسكي، و لاري شاودر، و البديل الغير خطي للاري شاودر، واخيرا كوفتز ونادلار. ولقد عرضنا بعض الامثلة للنظريات المثبتة لتوضيح طريقة تطبيقها على عينات تفاضلية افتراضية.

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Chapter 1

Introduction

Fractional differential equations may be used to describe various phenomenon in physics, control, engineering, etc. (see [10],[14],[24],[31],[35],[39],[49],[50],[53],[56],[57],[62], and references therein). Particularly, applications on antiperiodic fractional differential equations in physics have seen considerable developments in the last decade (see [15],[38] and references therein).

Most of fractional equations are obtained from a classical equation, like the wave equation or the diffusion equation, by replacing the time derivative by a fractional derivative. The fractional differential equations with initial and boundary conditions have gained much attention recently due to the fact that fractional order system response ultimately converges to the integer order system response. The most widely used definitions of an integral of fractional order is via an integral transform of a function f, called the Riemann–Liouville operator of fractional integration of order α and Caputo operator which is the closest one to reality due to the zero derivative of the constant. In this interpretation, the fractional derivative is left inverse of the fractional integral which is a natural generalization of the Cauchy formula for the n-fold primitive of a function f.

The natural question that will arise concerns the physical meaning of fractional integration. In ([56],[57]), the fractional integration can be considered as integration in some fractional dimensional space in the field of statistical mechanics and fractal media respectively. Whereas, in [49], it is shown that the geometric interpretation of fractional integration is "shadows on the walls" and its physical interpretation is "shadows of the past" whereas in [53], it was shown that there is a relation between stable probability distributions and the fractional integral.

Fractional integrals provide the language for formulating and analyzing many laws in physics and more particularly in astrophysics (stellar dynamics) where fractional kinetic equations play a crucial role [14]. Analytical solutions of fractional differential equations are of fundamental importance in describing and understanding dispersion phenomena, since all the parameters are expressed in a mathematically closed form and therefore the influence of individual parameters on a certain cause can be easily examined (see [24],[50]).

Several methods for solving fractional differential equations are presented in the literature. Analytical methods such as the Laplace transform method, the Fourier transform method , the Mellin transform method, the iteration method and the Green function method are presented (see [27],[40],[44],[48],[51]). Recently, other methods providing an analytic approximate solution, such as the adomian decomposition method [29], and homotopy perturbation method [46]. Numerical schemes for solving fractional differential equations are given in ([21],[22],[23],[45]).

If the initial condition is uncertain, it is preferable to make sure that the solution belongs to a known set. We have encountered a number of fractional differential equations, each of which has a solution and apparently only one solution. This raises the question of whether this is true of all fractional differential equations. In other words, does every fractional differential equation has exactly one solution? This may be an important question even for non-mathematicians. If you encounter any fractional differential equation in the science of investigating some physical problem, you might want to know that it has a solution before spending very much time and effort in trying to find it. Further, if you are successful in finding one solution, you might be interested in knowing whether you should continue a search for other possible solutions or whether you can be sure that there are no other solutions.

Existence and uniqueness of solutions for initial and boundary value problems as the main theoretical branch of fractional differential equations and inclusions have been studied by many authors ([1],[2],[9],[19],[20],[37],[54]). Usually, the existence and uniqueness of the solution for nonlinear fractional differential equations or inclusions are obtained by applying some fixed point theorems ([42],[43],[66],[67]), the classical method which is involved of comparison principle ([33],[34]), variational iteration and numerical approach ([11],[47]), Cauchy problem for evolution equations were discussed in ([63]-[65]), etc.

Some of boundary value problems for fractional differential equations or inclusions in the nonlocal sense are closest in modelling for some physical phenomena than the standard local boundary conditions (see [15], [24],[41]). Anti-periodic and integral boundary conditions for fractional differential equation or inclusions are classes of such kind of fractional differential models. Let us introduce some of these valuable works, and the reader can refer to ([2],[5],[6],[8] and references therein) for more other works.

Ahmed, et al. [4] studies the anti-periodic boundary value

$$\begin{cases} {}^cD_0^q x(t) = f(t, x(t)), t \in [0, T], \\ x^{(k)}(0) = -x^{(k)}(T),\ k = 0, 1, \end{cases}$$

of fractional order $1 < q \leq 2$.

Ahmed, et al. [3] studies the anti-periodic boundary value

$$\begin{cases} {}^cD_0^q x(t) = f(t, x(t)), t \in [0, T], \\ x^{(k)}(0) = -x^{(k)}(T),\ k = 0, 1, 2, \end{cases}$$

of fractional order $2 < q \leq 3$.

Agarwal, et al. [1] studies the anti-periodic boundary value

$$\begin{cases} {}^cD_0^q x(t) = f(t, x(t)), t \in [0, T], \\ x^{(k)}(0) = -x^{(k)}(T),\ k = 0, 1, 2, 3, 4, \end{cases}$$

of fractional order $3 < q \leq 4$.

Alsaedi, et al. [7] studies the anti-periodic boundary value

$$\begin{cases} {}^cD_0^q x(t) = f(t, x(t)), t \in [0, T], \\ x^{(k)}(0) = -x^{(k)}(T),\ k = 0, 1, 2, 3, 4, \end{cases}$$

of fractional order $4 < q \leq 5$.

In [60], Xu considered the following anti-periodic and integral fractional boundary value problem:

$$\begin{cases} {}^cD_0^q x(t) = f(t, x(t)), t \in [0, 1], q \in (1, 2] \\ x(1) = \mu \int_0^1 x(s) ds,\ x'(0) + x'(1) = 0. \end{cases}$$

The existence and uniqueness results in the above problems are obtained by applying some well known fixed point theorems such as Banach fixed point, Krasnoselskii, Schaefer, and the Leray-Schauder degree theorems for the case of single functions, and for the multivalued functions, we use Covitz and Nadler and Leray-Schauder nonlinear alternative fixed points theorems.

Motivated by these works, we study in the Thesis, the uniqueness and existence of solution for the fractional differential equations with anti-periodic and integral conditions involving the Caputo fractional derivative. We use the Caputo derivative because it is closed to nature and better for application than the other kinds of fractional derivatives.

This Thesis is organized as follows: Chapter 2 views the preliminaries that are needed in our studying for the existence and uniqueness problems.

Chapter 3 studies the existence of solutions for nonlinear fractional differential equation

$$^{c}D_{t_0}^{q}x(t) = f(t, x(t)), t \in J = [t_0, T], T > t_0, q \in (4, 5],$$

with anti-periodic and integral boundary conditions

$$x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^{T} g_k(t, x(t))dt, k = 0, 1, 2, 3, 4, \tag{1.1}$$

The theorems are illustrated by some examples.

Chapter 4 studies boundary value problems of fractional differential inclusion that have a form

$$^{c}D_{t_0}^{q}x(t) \in F(t, x(t)), t \in J, q \in (4, 5],$$

where $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, g_k,θ_k, and β_k are defined as above with anti-periodic type integral boundary condition (1.1). We end this Thesis by a conclusion in chapter 5.

Chapter 2

Preliminary Background

This chapter is focusing on introducing some facts and properties of fractional calculus and recalling some theorems from analysis that will be used for dealing with the nonlinear fractional differential equations.

2.1 Gamma Function

In this section, some of special functions are introduced that will be used in other chapters. We recall some information on the gamma function which plays the most important role in the theory of fractional differential equations. Undoubtedly, one of the basic functions of the fractional calculus is Euler's gamma function $\Gamma(z)$, which generalizes the factorial notation and allows n to be replaced by non-integer and even complex values [25].

Definition 2.1 *The gamma function* $\Gamma(z)$ *is defined by the integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

which converges in the right half of the complex plane.

By analytic continuation, gamma function can be extended to the whole complex plane except for the points $0, -1, -2, -3, \ldots$, where it has simple poles.

There are some formulas play an important role in various transformations and calculations involving $\Gamma(z)$, given by the following theorem.

Theorem 2.2 *[25]Let z be any complex number then,*
(a) $\Gamma(z+1) = z\Gamma(z)$,

(b) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$,

(c) $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt[2]{\pi}\Gamma(2z)$.

Corollary 2.3 *(a)* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

(b) $\Gamma(1) = 1$,

(c) $\Gamma(n+1) = n!$ *for any natural number* n.

[25]The gamma function is studied by many mathematicians. There is a long list of well-known definitions and theorems but the formulas in this section are sufficient.

2.2 Fractional Calculus

We recall in this section some facts from fractional calculus (see the monographs [30],[44],[48],[51]).

The Cauchy's formula for repeated integration

$$I_{t_0}^n f(t) = \int_{t_0}^{t} \int_{t_0}^{\tau_1} \dots \int_{t_0}^{\tau_{n-1}} f(\tau) d\tau \dots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_{t_0}^{t} f(\tau)(t-\tau)^{n-1} d\tau,$$

holds for $n \in \mathbb{N}$, $t \in \mathbb{R}$.

If n substituted by a positive real number q and $(n-1)!$ by it's generalization $\Gamma(q)$, a formula of fractional integration is obtained.

Definition 2.4 *[48]The Riemann–Liouville fractional integral of order q is defined as*

$$I_{t_0}^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(t) ds, q > 0,$$

provided the integral exists.

After the introduction of the fractional integration operator it is reasonable to define also the fractional differentiation operator. There are different definitions of fractional derivative, which do not coincide in general. The Caputo operator is the only fractional derivative used in this Thesis.

Definition 2.5 *[48]The Caputo derivative of a continuous function* $x : [t_0, \infty) \to \mathbb{R}$, *is defined as*

$${}^c D_{t_0}^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds, n-1 < q < n, n = [q]+1,$$

provided the fractional integral of the nth derivative $x^{(n)}$ exists. Here $[q]$ denotes the integer part of the real number q. This operator is introduced by the Italian mathematician Caputo in 1967.

Lemma 2.6 *[30]For $q>0$, the general solution of the fractional differential equation ${}^cD_{t_0}^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1},$$

and then

$$I_{t_0}^q {}^cD_{t_0}^q x(t) = x(t) + c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}$$

where $c_i \in \mathbb{R}$, $j = 0,1,2,...,n-1$, and $n = [q]+1$.

The fractional derivative operators and their related topics are studied by many mathematicians. There is a long list of well-known definitions and theorems but the formulas in this section are sufficient.

2.3 Fixed Point Theorem and Basic Concepts From Analysis

In this section, we present various fixed point theorems that will be used later on the sequel. We recall known definitions and notations from functional analysis (for more details see [17] [26],[52]).

Definition 2.7 *[26]A fixed point of a function is an element of the function's domain that is mapped to itself by the function. That is, c is a fixed point of the function f if and only if $f(c) = c$.*

Definition 2.8 *[52]A contraction mapping on a normed space $X = (X, \|\cdot\|)$ is a mapping T from X to itself, with the property that there is some nonnegative real number $0<k<1$ such that for all x and y in X,*

$$\|Tx - Ty\| \le k\|x-y\|.$$

Definition 2.9 *[52]Let Ω be an open bounded subset of a Banach spaces X, and the space of all continuous real valued functions on Ω denoted by $C(\Omega,\mathbb{R})$ be a Banach space endowed with the norm $\|x\| = \sup_{t\in\Omega}|x(t)|$. A subset K of $C(\Omega,\mathbb{R})$ is equicontinuous provided for every $\varepsilon>0$ there exists $\delta>0$ such that $\|x_1-x_2\|<\delta$ implies $\|f(x_1)-f(x_2)\|<\varepsilon$ for every $x_1,x_2\in\Omega$ and for every $f\in K$.*

Definition 2.10 *[52]A relatively compact subspace (or relatively compact subset) Y of a normed space X is a subset whose closure is compact.*

Remark 2.11 *[52]Every subset of a compact space is relatively compact since the closed subset of compact space is compact.*

Theorem 2.12 *(**Arzela-Ascoli Theorem**) Let Ω be an open bounded subset of a Banach spaces X. A subset of $C(\Omega,\mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

Theorem 2.13 *[17]Let X, and Y be Banach spaces. A mapping $T : X \to Y$ is compact if T_Y is relatively compact.*

Remark 2.14 *[52]A mapping $T : X \to Y$ is compact if T_Y is bounded and equicontinuous.*

Definition 2.15 *[26]Consider two Banach spaces X and Y, a subset Ω of X and a map $F : \Omega \to Y$. F is said to be completely continuous operator if it is continuous and maps bounded subsets of Ω into relatively compact sets.*

Next we introduce some fixed point theorems which will be used to prove the existence of solutions.

Theorem 2.16 *(Schaefer) Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $\Psi : \overline{\Omega} \to X$ be a completely continuous operator such that $\|\Psi x\| \leq \|x\|$, $x \in \partial\Omega$. Then Ψ has a fixed point in $\overline{\Omega}$.*

Theorem 2.17 *(Schauder) Let X be a Banach space. Assume that $\Psi : X \to X$ is completely continuous operator and the set $V = \{x \in X | x = \lambda \Psi x, 0 < \lambda < 1\}$ is bounded. Then Ψ has a fixed point in X.*

Theorem 2.18 *(Krasnoselskii) Let Ω be a closed convex and nonempty subset of a Banach space X. Let Φ, Θ be operators defined on Ω such that*

(i) $\Phi x + \Theta y \in \Omega$ whenever $x, y \in \Omega$;

(ii) Φ is compact and continuous;

(iii) Θ is a contraction mapping.

Then, there exists $z \in \Omega$ such that $z = \Phi z + \Theta z$.

The only theorem in this section that gives the uniqueness together with existence of a fixed point is the contraction mapping principle.

Theorem 2.19 ***(Banach)*** *Let X be a non-empty Banach space with a contraction mapping $T : X \to X$. Then T admits a unique fixed point in X.*

The fixed point theorems are studied by many mathematicians. There is a long list of fixed point theorems but the given fixed points in this section are sufficient.

Chapter 3

Existence of solution of Anti-periodic and Integral Boundary Value Problems for Fractional Differential Equations

3.1 Introduction

This chapter investigates the existence of solutions for fractional differential equations of order $q \in (4,5]$ with anti-periodic and integral boundary conditions by means of some standard fixed point theorems.

Precisely, we consider the problem:

$$\begin{cases} {}^{c}D_{t_0}^{q}x(t) = f(t,x(t)), t \in J, q \in (4,5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t,x(t))dt, k = 0,1,2,3,4, \end{cases} \tag{3.1}$$

where${}^{c}D_{t_0}^{q}$ denotes the Caputo fractional derivative of order q, f, $g_k : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and θ_k, $\beta_k \in \mathbb{R}$ with $\theta_k \neq 1$ for each $k = 0,1,2,3,4$.

This chapter is organized as follows. Section 2 views the preliminaries and develops the equivalent integral form to the problem (3.1). Section 3 studies the existence of solution for nonlinear fractional differential equations (3.1) with the given anti-periodic and integral boundary conditions. The theorems will be illustrated by some examples. Section 4 views the articles that can be produced from the problem (3.1) and describes the relationship with lower-order problems.

3.2 The Equivalent Integral Form for the Integral Boundary Value Problem

This section draws a basic result that is essential for the results in the sequel. To study the nonlinear problem (3.1), we need the following lemma.

Lemma 3.1 *For any $y \in C(J,\mathbb{R})$, the solution of the fractional linear boundary value problem*

$$\begin{cases} {}^{c}D_{t_0}^{q}x(t) = y(t), t \in J, 4 < q \leq 5, \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t)dt, \ k = 0,1,2,3,4, \end{cases} \tag{3.2}$$

is

$$\begin{aligned} x(t) &= \int\limits_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int\limits_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} y(s)\,ds \\ &\quad + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int\limits_{t_0}^{T} g_k(s)ds \end{aligned} \tag{3.3}$$

where

$$\alpha_k = \prod_{m=0}^{k} (1-\theta_m), \ \lambda_k(t) = \sum_{m=0}^{k} \gamma_{m,k} \begin{pmatrix} k \\ m \end{pmatrix} (t-t_0)^m (T-t_0)^{k-m}, k = 0,1,2,3,4,$$

and

$$\begin{aligned} \gamma_{0,0} &= 1, \gamma_{1,1} = \alpha_0, \gamma_{2,2} = \alpha_1, \gamma_{3,3} = \alpha_2,, \gamma_{4,4} = \alpha_3 \\ \gamma_{0,1} &= \theta_0, \gamma_{0,2} = \theta_0(\theta_1+1), \gamma_{0,3} = \theta_0(\theta_1\theta_2 + 2\theta_1 + 2\theta_2 + 1), \\ \gamma_{0,4} &= \theta_0(\theta_1\theta_2\theta_3 + 3\theta_1\theta_2 + 5\theta_1\theta_3 + 3\theta_2\theta_3 + 3\theta_1 + 3\theta_3 + 1), \\ \gamma_{1,2} &= 2\alpha_0\theta_1, \gamma_{1,3} = 3\alpha_0\theta_1(\theta_2+1), \gamma_{1,4} = 4\alpha_0\theta_1(\theta_2\theta_3 + 2\theta_2 + 2\theta_3 + 1), \\ \gamma_{2,3} &= 3\alpha_1\theta_2, \gamma_{2,4} = 6\theta_2\alpha_1(\theta_3+1), \gamma_{3,4} = 4\theta_3\alpha_2. \end{aligned}$$

Proof. Using Lemma 2.6, for some constants $c_0, c_1, c_2.c_3, c_4 \in \mathbb{R}$, we have

$$\begin{aligned} x(t) &= I_{t_0}^{q} y(t) - c_0 - c_1(t-t_0) - c_2(t-t_0)^2 - c_3(t-t_0)^3 - c_4(t-t_0)^4 \\ &= \int\limits_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds \\ &\quad -c_0 - c_1(t-t_0) - c_2(t-t_0)^2 - c_3(t-t_0)^3 - c_4(t-t_0)^4. \end{aligned} \tag{3.4}$$

Using the Libeniz rule for differentiation, we get

$$\begin{aligned}
x'(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds - c_1 - 2c_2(t-t_0) - 3c_3(t-t_0)^2 - 4c_4(t-t_0)^3, \\
x''(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds - 2c_2 - 6c_3(t-t_0) - 12c_4(t-t_0)^2, \\
x'''(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-4}}{\Gamma(q-3)} y(s)\,ds - 6c_3 - 24c_4(t-t_0), \\
x''''(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-5}}{\Gamma(q-4)} y(s)\,ds - 24c_4. \qquad (3.5)
\end{aligned}$$

Applying the boundary conditions of the problem (3.2) in $x(t), x'(t), x''(t), x'''(t), x''''(t)$, we have

$$\begin{aligned}
-c_0 &= \theta_0 x(T) + \beta_0 \int_{t_0}^{T} g_0(t)dt, \\
-c_1 &= \theta_1 x'(T) + \beta_1 \int_{t_0}^{T} g_1(t)dt, \\
-2c_2 &= \theta_2 x''(T) + \beta_2 \int_{t_0}^{T} g_2(t)dt, \\
-6c_3 &= \theta_3 x'''(T) + \beta_3 \int_{t_0}^{T} g_4(t)dt, \\
-24c_4 &= \theta_4 x''''(T) + \beta_4 \int_{t_0}^{T} g_4(t)dt, \qquad (3.6)
\end{aligned}$$

where

$$
\begin{aligned}
x(T) &= \int_{t_0}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)\,ds \\
&\quad -c_0 - c_1(T-t_0) - c_2(T-t_0)^2 - c_3(T-t_0)^3 - c_4(T-t_0)^4 \\
x^{'}(T) &= \int_{t_0}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds - c_1 - 2c_2(T-t_0) - 3c_3(T-t_0)^2 - 4c_4(T-t_0)^3, \\
x^{''}(T) &= \int_{t_0}^{T} \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds - 2c_2 - 6c_3(T-t_0) - 12c_4(T-t_0)^2, \\
x^{'''}(T) &= \int_{t_0}^{T} \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\,ds - 6c_3 - 24c_4(T-t_0), \\
x^{''''}(T) &= \int_{t_0}^{T} \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\,ds - 24c_4.
\end{aligned}
$$

Solving the equations in the system 3.6 for the unknowns c_0, c_1, c_2, c_3 and c_4 , we find that

$$
\begin{aligned}
c_0 \;=\; & \frac{-\theta_0}{(1-\theta_0)} \int_{t_0}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)\,ds - \frac{\theta_0\theta_1 (T-t_0)}{(1-\theta_0)(1-\theta_1)} \int_{t_0}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds \\
& - \frac{\theta_0\theta_2(\theta_1+1)(T-t_0)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)} \int_{t_0}^{T} \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds \\
& - \frac{\theta_0\theta_3(\theta_1\theta_2+2\theta_1+2\theta_2+1)(T-t_0)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)} \int_{t_0}^{T} \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\,ds \\
& - \frac{\theta_0\theta_4(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1)(T-t_0)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)} \times \\
& \int_{t_0}^{T} \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\,ds - \frac{\beta_0}{(1-\theta_0)} \int_{t_0}^{T} g_0(s,x(s))ds \\
& - \frac{\theta_0\beta_1(T-t_0)}{(1-\theta_0)(1-\theta_1)} \int_{t_0}^{T} g_1(s,x(s))ds \\
& - \frac{\theta_0\beta_2(\theta_1+1)(T-t_0)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)} \int_{t_0}^{T} g_2(s,x(s))ds \\
& - \frac{\theta_0\beta_3(\theta_1\theta_2+2\theta_1+2\theta_2+1)(T-t_0)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)} \int_{t_0}^{T} g_3(s,x(s))ds \\
& - \frac{\theta_0\beta_4(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1)(T-t_0)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)} \times \\
& \int_{t_0}^{T} g_4(s,x(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
c_1 \;=\; & -\frac{\theta_1}{(1-\theta_1)} \int_{t_0}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds - \frac{\theta_1\theta_2 (T-t_0)}{(1-\theta_1)(1-\theta_2)} \int_{t_0}^{T} \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds \\
& - \frac{\theta_1\theta_3(\theta_2+1)(T-t_0)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)} \int_{t_0}^{T} \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\,ds \\
& - \frac{\theta_1\theta_4(\theta_2\theta_3+2\theta_2+2\theta_3+1)(T-t_0)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)} \int_{t_0}^{T} \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\beta_1}{(1-\theta_1)}\int_{t_0}^{T} g_1(s,x(s))ds-\frac{\theta_1\beta_2\,(T-t_0)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds\\
&-\frac{\theta_1\beta_3(\theta_2+1)\,(T-t_0)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds\\
&-\frac{\theta_1\beta_4\,(\theta_2\theta_3+2\theta_2+2\theta_3+1)\,(T-t_0)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
c_2 \;=\; &-\frac{\theta_2}{2(1-\theta_2)}\int_{t_0}^{T}\frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)\,ds-\frac{\theta_2\theta_3\,(T-t_0)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)\,ds\\
&-\frac{\theta_2\theta_4\,(\theta_3+1)\,(T-t_0)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds\\
&-\frac{\beta_2}{2(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds-\frac{\theta_2\beta_3\,(T-t_0)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds\\
&-\frac{\theta_2\beta_4\,(\theta_3+1)\,(T-t_0)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
c_3 \;=\; &-\frac{\theta_3}{6(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)\,ds-\frac{\theta_3\theta_4\,(T-t_0)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds\\
&-\frac{\beta_3}{6(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds-\frac{\theta_3\beta_4\,(T-t_0)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,
\end{aligned}
$$

and

$$
c_4=-\frac{\theta_4}{24(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds-\frac{\beta_4}{24(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds.
$$

Substituting the values of c_0, c_1, c_2, c_3 and c_4 in (3.4), and arranging the terms into compact expression, we get (3.3). ■

3.3 Existence and Uniqueness Results

Let $C(J,\mathbb{R})$ denotes the Banach space of all real valued continuous functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t\in J\}$.

In view of Lemma 3.1, define an operator $\Psi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as

$$
\begin{aligned}
(\Psi x)(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s,x(s))ds \\
&\quad + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \qquad (3.7)
\end{aligned}
$$

Observe that problem (3.1) has a solution $x\in C(J,\mathbb{R})$ if and only if (3.7) satisfies the fixed point equation $\Psi x = x$.

Before going on to first result, the following hypothesis is essential.

(A) Let $L_k, k = 0,1,2,3,4,5$, be positive constants such that $|g_k(t,x(t))| \le L_k$, and $|f(t,x(t))| \le L_5$, for $t\in J,\ x\in C(J,\mathbb{R})$.

Lemma 3.2 *Assume that hypothesis (A) holds. Then, the operator Ψ is completely continuous.*

Proof. The continuity of g_k and f imply the continuity of the operator Ψ. By virtue of (3.7), we have

$$
\begin{aligned}
|(\Psi x)(t)| &\le \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&\quad + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s,x(s))| ds \\
&\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))| ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{L_5(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{L_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k\,|\beta_k|\,(T-t_0) \right) \\
&\le \max_{t\in J} \left(\frac{L_5(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{L_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k\,|\beta_k|\,(T-t_0) \right) \right) \\
&= L,
\end{aligned}
$$

which implies that $\|\Psi x\| \leq L$. Furthermore,

$$
\begin{aligned}
&\left|(\Psi x)^{'}(t)\right| \\
\leq\ & \int_{t_0}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))| ds \\
&+\sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} \left|\lambda_k^{'}(t)\right| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s,x(s))|\, ds \\
&+\sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} \left|\lambda_k^{'}(t)\right| \int_{t_0}^{T} |g_k(s,x(s))|\, ds \\
\leq\ & \frac{L_5(t-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\left|\lambda_k^{'}(t)\right|}{k!\,|\alpha_k|} \left(\frac{L_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k\,|\beta_k|\,(T-t_0) \right) \\
\leq\ & \max_{t\in J} \left(\frac{L_5(t-t_0)^{q}}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\left|\lambda_k^{'}(t)\right|}{k!\,|\alpha_k|} \left(\frac{L_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k\,|\beta_k|\,(T-t_0) \right) \right) \\
=\ & L^{'}
\end{aligned}
$$

and this implies that $\left\|(\Psi x)^{'}\right\| \leq L$. Hence, for $t_1,\ t_2 \in J$, we have

$$|(\Psi x)(t_2)-(\Psi x)(t_1)| \leq \int_{t_1}^{t_2} \left|(\Psi x)^{'}(s)\right| ds \leq L^{'}(t_2-t_1).$$

This implies the equicontinuity of Ψ on J. Thus, by the Arzela-Ascoli theorem, the operator Ψ is completely continuous. This finishes the proof. ■

To establish the first existence result based on the fixed point Theorem (2.16), we need the following assumption.

(B) Let σ, τ and ρ be positive constants such that

$$
\begin{cases}
\tau = \max_{t\in J} \left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum\limits_{k=0}^{4} \frac{|\lambda_k(t)|}{k!|\alpha_k|} \left(\frac{|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k|\,(T-t_0) \right) \right), \sigma\tau < 1, \\
|f(t,x(t))| \leq \sigma\,|x(t)|, \\
|g_k(t,x)| \leq \sigma\,|x(t)|, k=0,1,2,3,4,
\end{cases}
$$

for $|x(t)| < \rho, t \in J$.

Notice that assumption (A) can be followed by assumption (B).

Theorem 3.3 *Assume that hypothesis (B) holds. Then, the problem (3.1) has at least one solution.*

Proof. Define a bounded nonempty open subset $\Omega = \{x \in C(J,\mathbb{R}) : \|x\| < \rho\}$. Then, by Lemma 3.2, the operator $\Psi : \overline{\Omega} \to C(J,\mathbb{R})$ is completely continuous and satisfying

$$|(\Psi x)(t)| \leq \sigma \left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k|\,(T-t_0) \right) \right) \|x\|. \quad (3.8)$$

In accordance with hypothesis (B), Equation (3.8) imply

$$\|\Psi x\| \leq \|x\|,\ x \in \partial\Omega.$$

Hence, by Theorem 2.16, Ψ has at least one fixed point which is a solution of the problem (3.1). ■

Theorem 3.4 *If alternatively, assuming that*

$$\lim_{x\to 0} \frac{f(t,x)}{x} = \lim_{x\to 0} \frac{g_k(t,x)}{x} = 0,$$

and

$$\varepsilon \max_{t\in J} \left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k|\,(T-t_0) \right) \right) \leq 1,$$

for any positive constant $\varepsilon > 0$, we have the same result of Theorem 3.3.

Assuming hypothesis (A), and letting $x \in C(J,\mathbb{R})$ such that $x = \lambda \Psi x$ for $0 < \lambda < 1$. Then, we have $\|x\| \leq \|\Psi x\| \leq L$. Hence, by virtue of Theorem 2.17, the following result follows.

Theorem 3.5 *Let Ψ be defined as in (3.7). If assumption (A) holds, then the problem (3.1) has a solution.*

Our next existence result is based on Krasnoselskii's fixed point Theorem 2.18, which needs the following assumption.

(C) Let $C_k \in \mathbb{R}^+$, $k = 0,1,2,3,4,5$, such that

$$\begin{cases} |g_k(t,x(t)) - g_k(t,y(t))| \leq C_k\,|x(t)-y(t)|\,, k = 0,1,2,3,4, \\ |f(t,x(t)) - f(t,y(t))| \leq C_5\,|x(t)-y(t)|\,, \end{cases}$$

for $t \in J, x, y \in C(J,\mathbb{R})$.

(D) Let $\mu_k \in C(J,\mathbb{R})$, $k = 0,1,2,3,4,5$, such that

$$\begin{cases} g_k(t,x(t)) \le \mu_k(t), k = 0,1,2,3,4, \\ f(t,x(t)) \le \mu_5(t), \end{cases}$$

for $t \in J, x, y \in C(J,\mathbb{R})$.

Theorem 3.6 *Let $f : J \times C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be jointly continuous function. If assumptions (C) and (D) hold, then the problem (3.1) has a solution if*

$$\sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|} \left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + C_k\,|\beta_k|\,(T-t_0) \right) < 1. \tag{3.9}$$

Proof. Let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$, for some fixed positive constant r that satisfying

$$r \ge \frac{\|\mu_5\|\,(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|} \left(\frac{\|\mu_5\|\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + \|\mu_k\|\,|\beta_k|\,(T-t_0) \right).$$

Define the operators Φ and Θ on B_r as

$$(\Phi x)(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))\,ds,$$

and

$$(\Theta x)(t) = \sum_{k=0}^{4} \frac{\lambda_k(t)}{k!\alpha_k} \left(\theta_k \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s,x(s))ds + \beta_k \int_{t_0}^{T} g_k(s,x(s))ds \right).$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Theta y\| \le r.$$

Thus, $\Phi x + \Theta y \in B_r$. Moreover, if $x, y \in B_r$, then

$$\begin{aligned} &|(\Theta y)(t) - (\Theta x)(t)| \\ \le\ & \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(|\theta_k| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} \,|f(s,x(s))|\,ds + |\beta_k| \int_{t_0}^{T} |g_k(s,x(s))|\,ds \right). \end{aligned}$$

In accordance with (3.9), Θ is a contraction mapping on B_r.

Continuity of f implies the continuity of Φ. Also, Φ is uniformly bounded on B_r as $\|\Phi x\| \le \frac{\|\mu_5\|(T-t_0)^q}{\Gamma(q+1)}$. Next, we prove the compactness of Φ.

Let $\sup_{(t,x)\in J\times Br}\|f(t,x)\| = f^* < \infty$, then, for $t_1, t_2 \in J$, we have

$$\begin{aligned}\|(\Phi x)(t_2)-(\Phi x)(t_1)\| &= \frac{1}{\Gamma(q)}\left\|\int_{t_0}^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]f(s,x(s))\,ds\right. \\ &\quad \left.+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))\,ds\right\| \\ &\leq \frac{f^*}{\Gamma(q+1)}\left(2|t_2-t_1|^q+|(t_2-t_0)^q-(t_1-t_0)^q|\right),\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. So Φ is relatively compact on B_r. Hence, by the Arzela-Ascoli theorem, Φ is compact on B_r. Thus all assumptions of Theorem 2.18 are satisfied. Therefore, the problem (3.1) has a solution. This completes the proof. ■

The existence and uniqueness result can be obtained by the well-known Banach fixed point theorem.

Theorem 3.7 *Let $f, g_k : J\times C(J,\mathbb{R}) \to C(J,\mathbb{R}), k = 0,1,2,3,4,$ be jointly continuous functions satisfying the hypothesis (C). Then the problem (3.1) has a unique solution if*

$$\gamma = \frac{C_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)} + C_k|\beta_k|(T-t_0)\right) < 1.$$

Proof. Setting $\sup_{t\in J}|g_k(t,0)| = M_k$, and $\sup_{t\in J}|f(t,0)| = M_5$, and selecting

$$r \geq \frac{\frac{M_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!|\alpha_k|}\left(\frac{M_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)} + M_k|\beta_k|(T-t_0)\right)}{1-\gamma}.$$

If $x \in B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$, then

$$\begin{aligned}
|(\Psi x)(t)| &\leq \max_{t\in J}\left\{\int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\right.\\
&\quad+\sum_{k=0}^{4}\frac{|\theta_k|}{k!\,|\alpha_k|}|\lambda_k(t)|\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\\
&\quad\left.+\sum_{k=0}^{4}\frac{|\beta_k|}{k!\,|\alpha_k|}|\lambda_k(t)|\int_{t_0}^{T}(|g_k(s,x(s))-g_k(s,0)|+|g_k(s,0)|)ds\right\}\\
&\leq \frac{M_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{M_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+M_k\,|\beta_k|\,(T-t_0)\right)\\
&\quad+\left(\frac{C_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right)\right)r\\
&\leq (1-\gamma)\,r+\gamma r = r.
\end{aligned}$$

Hence Ψ maps the subset B_r into itself. Now, for $x,y \in B_r$, and $t \in J$, we obtain

$$\begin{aligned}
&|(\Psi x)(t)-(\Psi y)(t)|\\
\leq\;& \max_{t\in J}\left\{\int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|\,ds\right.\\
&+\sum_{k=0}^{4}\frac{|\theta_k|}{k!\,|\alpha_k|}|\lambda_k(t)|\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}|f(s,x(s))-f(s,y(s))|\,ds\\
&\left.+\sum_{k=0}^{4}\frac{|\beta_k|}{k!\,|\alpha_k|}|\lambda_k(t)|\int_{t_0}^{T}|g_k(s,x(s))-g_k(s,y(s))|\,ds\right\}\\
\leq\;& \max_{t\in J}\left(\frac{C_5(t-t_0)^q}{\Gamma(q+1)}ds+\sum_{k=0}^{4}\frac{|\lambda_k(t)|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right)\right)\|x-y\|\\
\leq\;& \gamma\|x-y\|.
\end{aligned}$$

Since γ depends only upon the parameters involved in the problem. Therefore, Ψ is a contraction operator. Hence, the conclusion of the theorem follows by the Banach fixed point theorem. This finishes the proof. ■

The last result of existence problems is due to the Leray-Schauder degree theorem. The following hypothesis is sufficient for the next theorem.

(E) Let $A_k, B_k, k = 0,1,2,3,4,5$, be positive constants satisfying

$$\begin{cases} |g_k(t,x(t))| \le A_k |x(t)| + B_k, k = 0,1,2,3,4 \\ |f(t,x(t))| \le A_5 |x(t)| + B_5 \end{cases}$$

for $t \in J, x \in C(J,\mathbb{R})$. Moreover, assume

$$\begin{cases} B = \frac{B_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!|\alpha_k|} \left(\frac{B_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)} + B_k |\beta_k| (T-t_0) \right), \\ A = \frac{A_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!|\alpha_k|} \left(\frac{A_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)} + A_k |\beta_k| (T-t_0) \right) < 1. \end{cases}$$

Theorem 3.8 *Let $f, g_k : J \times C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be jointly continuous functions. If hypothesis (E) holds, then the problem (3.1) has at least one solution.*

Proof. Define a fixed point problem by

$$x = \Psi x, \tag{3.10}$$

where Ψ is defined by (3.7). Then we only need to prove the existence of at least one solution $x \in C(J,\mathbb{R})$ satisfying (3.10). Define a suitable ball $B_r \subset C(J,\mathbb{R})$ with radius $r > 0$ as $B_r = \{x \in C(J,\mathbb{R}) : \|x\| < r\}$, where r will be fixed later. Then, it is sufficient to show that $\Psi : \overline{B_r} \to C(J,\mathbb{R})$ satisfies

$$0 \notin (I - \lambda\Psi)(\partial B_r) \tag{3.11}$$

for any $\lambda \in [0,1]$. Here I denotes the identity operator. Let us define the homotopy

$$h_\lambda(x) = x - \lambda\Psi x, x \in X, \ \lambda \in [0,1].$$

Then, following the same steps of proof Lemma 3.2, $h_\lambda = I - \lambda\Psi$ is completely continuous. Using condition (3.11), and the homotopy invariance property of Leray-Schauder degrees, we have

$$\begin{aligned} deg(h_\lambda, B_r, 0) &= deg((I - \lambda\Psi), B_r, 0) = deg(h_1, B_r, 0) \\ &= deg(h_0, B_r, 0) = deg(I, B_r, 0) = 1 \neq 0,, \ 0 \in B_r. \end{aligned}$$

Hence, there is at least one $x \in B_r$, such that equation (3.10) is true. It remains to find the constant r satisfying (3.11). Therefore, for any $t \in J$, and $x \in B_r$ satisfying $x = \lambda\Psi x$ for some $\lambda \in [0,1]$, we have

$$\begin{aligned}
|x(t)| &= |\lambda \Psi x(t)| \\
&\leq \frac{B_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|} \left(\frac{B_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + B_k\,|\beta_k|\,(T-t_0) \right) \\
&\quad \left(\frac{A_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|} \left(\frac{A_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + A_k\,|\beta_k|\,(T-t_0) \right) \right) \|x\| \\
&= B + A\,\|x\|\,.
\end{aligned}$$

Hence

$$\|x\| \leq \frac{B}{1-A}.$$

Choosing $r > \frac{B}{1-A}$, then equation (3.11) holds. This completes the proof. ■

Example 3.9 *Consider the problem*

$$\begin{cases} {}^{c}D_{1}^{4.2}x(t) = \frac{1}{(e^t+2)^3}\frac{|x(t)|}{|x(t)|+1}, t \in [1,2], \\ x^{(k)}(1) + x^{(k)}(2) = \frac{1}{2^{k+1}} \int\limits_{1}^{2} \frac{\sin x(t)}{(t+2)^3} ds, k = 0,1,2,3,4. \end{cases} \tag{3.12}$$

where x is a real valued function defined on $J = [1,2]$. In accordance with the hypotheses in 3.2, we obtain

$$\begin{aligned}
\left| \frac{1}{(e^t+2)^3} \frac{|x(t)|}{|x(t)|+1} \right| &= \left| \frac{1}{(e^t+2)^3} \left(1 - \frac{1}{|x(t)|+1} \right) \right| \\
&\leq \left| \frac{1}{(e^t+2)^3} \right| \\
&\leq \left| \frac{1}{2^3} \right| = \frac{1}{8}.
\end{aligned}$$

Since $|\sin x(t)| \leq 1$ for all $t \in J = [1,2]$, then

$$\begin{aligned}
\left| \frac{\sin x(t)}{(t+2)^3} \right| &\leq \left| \frac{1}{(t+2)^3} \right| \\
&\leq \left| \frac{1}{2^3} \right| = \frac{1}{8}.
\end{aligned}$$

Thus we can choose

$$L_k = \frac{1}{8}.$$

Also

$$\begin{aligned}\left|\frac{1}{(e^t+2)^3}\frac{|x(t)|}{|x(t)|+1}\right| &= |x(t)|\left|\frac{1}{(e^t+2)^3}\frac{1}{|x(t)|+1}\right| \\ &\leq |x(t)|\left|\frac{1}{(e^t+2)^3}\right| \\ &\leq |x(t)|\left|\frac{1}{2^3}\right| = |x(t)|\frac{1}{8},\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\sin x(t)}{(t+2)^3}\right| &= |x(t)|\left|\frac{\sin x(t)}{|x(t)|\,(t+2)^3}\right| \\ &\leq |x(t)|\left|\frac{\sin x(t)}{(t+2)^3}\right| \\ &\leq |x(t)|\left|\frac{1}{(t+2)^3}\right| \\ &\leq |x(t)|\left|\frac{1}{2^3}\right| = |x(t)|\frac{1}{8}.\end{aligned}$$

So we can take $\sigma = \frac{1}{8}$.

Now

$$\begin{aligned}\tau &= \max_{t\in J}\left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4}\frac{|\lambda_k(t)|}{k!\,|\alpha_k|}\left(\frac{|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k|\,(T-t_0)\right)\right) \\ &= \frac{(2-1)^{4.2}}{\Gamma(4.2+1)} + \sum_{k=0}^{4}\frac{|\lambda_k(2)|}{k!\,|\alpha_k|}\left(\frac{|-1|\,(2-1)^{4.2-k}}{\Gamma(4.2-k+1)} + \left|\frac{1}{2^{k+1}}\right|(2-1)\right) \\ &= 0.665.\end{aligned}$$

Note that $\sigma\tau < 1$. *Thus all hypotheses in 3.3 are hold.*

$$\tau = 0.665 < 1, A = \gamma = \frac{1}{8}\tau < 1.$$

On the other hand, we have

$$\begin{aligned}
&\left|\frac{1}{(e^t+2)^3}\frac{|x(t)|}{|x(t)|+1}-\frac{1}{(e^t+2)^3}\frac{|y(t)|}{|y(t)|+1}\right| \\
=&\left|\frac{1}{(e^t+2)^3}\left\{\left(1-\frac{1}{|x(t)|+1}\right)-\left(1-\frac{1}{|y(t)|+1}\right)\right\}\right| \\
=&\left|\frac{1}{(e^t+2)^3}\left(\frac{1}{|y(t)|+1}-\frac{1}{|x(t)|+1}\right)\right| \\
=&\left|\frac{|x(t)-y(t)|}{(e^t+2)^3}\left(\frac{|x(t)|-|y(t)|}{|x(t)-y(t)|\,(|x(t)|+1)\,(|y(t)|+1)}\right)\right| \\
\leq&\ |x(t)-y(t)|\left|\frac{1}{(e^t+2)^3}\left(\frac{1}{(|x(t)|+1)\,(|y(t)|+1)}\right)\right| \\
\leq&\ |x(t)-y(t)|\left|\frac{1}{(e^t+2)^3}\right| \\
\leq&\ |x(t)-y(t)|\left|\frac{1}{(2)^3}\right|=|x(t)-y(t)|\,\frac{1}{8},
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\sin x(t)}{(t+2)^3}-\frac{\sin x(t)}{(t+2)^3}\right| &= \left|\frac{1}{(t+2)^3}\left(\sin x(t)-\sin y(t)\right)\right| \\
&= |x(t)-y(t)|\left|\frac{1}{(t+2)^3}\frac{(\sin x(t)-\sin y(t))}{|x(t)-y(t)|}\right| \\
&\leq |x(t)-y(t)|\left|\frac{1}{(t+2)^3}\left(\sin x(t)-\sin y(t)\right)\right| \\
&\leq |x(t)-y(t)|\left|\frac{1}{(2)^3}\right|=|x(t)-y(t)|\,\frac{1}{8}.
\end{aligned}$$

So we can choose $C_k=\frac{1}{8}$ *for the hypotheses in 3.5.Thus all hypotheses in 3.5 are hold.*

Observe that

$$\frac{\sin x(t)}{(t+2)^3}\leq\frac{1}{(t+2)^3}$$

for all $t\in J=[1,2]$, *and*

$$\frac{1}{(e^t+2)^3}\leq\frac{1}{(t+2)^3}$$

for all $t\in J=[1,2]$. *hence we can choose*

$$\mu_k=\frac{1}{(t+2)^3},k=0,1,2,3,4,\mu_5=\frac{1}{(t+2)^3}.$$

for the hypotheses in 3.6.Thus all hypotheses in 3.6 are hold since

$$\begin{aligned}
& \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right) \\
= \; & \sum_{k=0}^{4} \frac{\|\lambda_k(2)\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|-1|\,(2-1)^{4.2-k}}{\Gamma(4.2-k+1)}+C_k\left|\frac{1}{2^{k+1}}\right|(2-1)\right) \\
\leq \; & \sum_{k=0}^{4} \frac{\|\lambda_k(2)\|}{k!\,|\alpha_k|}\left(\frac{\frac{1}{8}\,|-1|\,(2-1)^{4.2-k}}{\Gamma(4.2-k+1)}+\frac{1}{8}\left|\frac{1}{2^{k+1}}\right|(2-1)\right) \\
= \; & \frac{1}{8}\sum_{k=0}^{4} \frac{\|\lambda_k(2)\|}{k!\,|\alpha_k|}\left(\frac{|-1|\,(2-1)^{4.2-k}}{\Gamma(4.2-k+1)}+\left|\frac{1}{2^{k+1}}\right|(2-1)\right) \\
= \; & \frac{1}{8}\tau<1.
\end{aligned}$$

So we have

$$\begin{cases} L_k = C_k = \sigma = \frac{1}{8}, \\ \mu_k = \frac{1}{(t+2)^3}, k=0,1,2,3,4, \mu_5 = \frac{1}{(t+2)^3} \\ \tau = 0.665 < 1, \gamma = \frac{1}{8}\tau < 1. \end{cases}$$

Hence, using any of above theorems, the existence of solution for the problem (3.12) can be obtained.

The new existence results for a class of fifth-order nonlinear differential equations with anti-periodic boundary conditions can be obtained as a special case by taking $q = 5$ in the results of this section.

3.4 Relationship with Lower-Order Problems

This section views the articles that can be produced from the results of previous section and describes the relationship with lower-order problems. Here, we assume that $t_0 = 0$.

We observe that when we put $\theta_0 = -1$ and $\beta_0 = 0$, the first term in expressions for $x(t)$ in (3.3) corresponds to the anti periodic boundary value problem (see [2])

$$\begin{cases} {}^cD_0^q x(t) = f(t, x(t)), t \in [0,T], 0 < q \leq 1, \\ x(0) = -x(T). \end{cases}$$

When we put $\theta_0 = -1 = \theta_1$ and $\beta_0 = 0 = \beta_1$, the first two terms in the reduced expression for $x(t)$ correspond to the anti periodic boundary value problem (see [4])

$$\begin{cases} {}^cD_0^q x(t) = f(t,x(t)), t \in [0,T], 1 < q \leq 2, \\ x^{(k)}(0) - \theta_k x^{(k)}(T) = 0,\ k = 0,1. \end{cases}$$

When we put $\theta_0 = \theta_1 = \theta_2 = -1$ and $\beta_0 = \beta_1 = \beta_2 = 0$, the first three terms in the reduced expression for $x(t)$ correspond to the anti periodic boundary value problem (see [3])

$$\begin{cases} {}^cD_0^q x(t) = f(t,x(t)), t \in [0,T], 2 < q \leq 3, \\ x^{(k)}(0) - \theta_k x^{(k)}(T) = 0,\ k = 0,1,2. \end{cases}$$

When we put $\theta_0 = \theta_1 = \theta_2 = \theta_3 = -1$ and $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$, the first four terms in the reduced expression for $x(t)$ correspond to the anti periodic boundary value problem (see [1])

$$\begin{cases} {}^cD_0^q x(t) = f(t,x(t)), t \in [0,T], 3 < q \leq 4, \\ x^{(k)}(0) - \theta_k x^{(k)}(T) = 0,\ k = 0,1,2,3. \end{cases}$$

When we put $\theta_0 = \theta_1 = \theta_2 = \theta_3 = \theta_4 = -1$ and $\beta_0 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, the first five terms in the reduced expressions for $x(t)$ corresponds to the anti periodic boundary value problem (see [7])

$$\begin{cases} {}^cD_0^q x(t) = f(t,x(t)), t \in [0,T], 4 < q \leq 5, \\ x^{(k)}(0) - \theta_k x^{(k)}(T) = 0,\ k = 0,1,2,3,4. \end{cases}$$

The first two terms and the term seven in the expression for $x(t)$ correspond to the anti periodic boundary value problem (see [5])

$$\begin{cases} {}^cD_0^q x(t) = f(t,x(t)),\ t \in [0,T], T > 0,\ q \in (0,1] \\ x(0) - \theta x(T) = \beta \int_0^T g(t,x(t))dt. \end{cases}$$

The first two terms and the terms seventh and eighth in the expression for $x(t)$ correspond to the anti periodic boundary value problem (see [6])

$$\begin{cases} {}^cD_0^q x(t) = f(t,x(t)),\ t \in [0,T], T > 0,\ q \in (1,2] \\ x^{(k)}(0) - \theta_k x^{(k)}(T) = \beta_k \int_0^T g_k(t,x(t))dt,\ k = 0,1. \end{cases}$$

The first three terms and the terms seventh, eighth and ninth in the expression for $x(t)$ correspond to the anti periodic boundary value problem (see [9])

$$\begin{cases} {}^{c}D_{0}^{q}x(t) = f(t,x(t)),\ t \in [0,T], T > 0,\ q \in (2,3] \\ x^{(k)}(0) - \theta_{k}x^{(k)}(T) = \beta_{k}\int\limits_{0}^{T} g_{k}(t,x(t))dt,\ k = 0,1,2. \end{cases}$$

From the above deductions, it can easily be concluded that expressions of $x(t)$ for an anti-periodic boundary value problem of fractional order $q \in (4,5]$ contains solution for lower-order fractional anti-periodic problems. We can further interpret that the last term in expressions of $x(t)$ arises due to consideration of the order $q \in (4,5]$, whereas the remaining terms correspond to the lower-order problems. This observation gives a useful insight into the study of anti-periodic fractional boundary value problems that a unit-increase in the fractional order of the problem gives rise to a new term in expressions for $x(t)$, preserving the terms corresponding to lower-order anti-periodic problems. In other words, one can say that solution for a higher-order anti-periodic fractional boundary value problem inherits all the characteristics of lower-order fractional anti-periodic problems. Hence, our results generalize the existing results on anti-periodic fractional boundary value problems.

Chapter 4

Existence for Anti-periodic and Integral Boundary Value Problems of Fractional Differential Inclusions

4.1 Introduction

This chapter investigates the existence of solutions for fractional differential inclusions of order $q \in (4,5]$ with anti-periodic and integral boundary conditions by means of some standard fixed point theorems for inclusions. Our results include the cases when the multivalued map involved in the problem has convex as well as non-convex values. The classical integer-order differential inclusions have been considered extensively by the researchers (see for example [18] and the references therein).

Consider the following inclusion problem:

$$\begin{cases} {}^cD_{t_0}^q x(t) \in F(t, x(t)), t \in J = [t_0, T], T > t_0, q \in (4,5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t, x(t))dt, k = 0,1,2,3,4, \end{cases} \tag{4.1}$$

where ${}^cD_{t_0}^q$ denotes the Caputo fractional derivative of order q, $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $g_k : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and θ_k, $\beta_k \in \mathbb{R}$ with $\theta_k \neq 1$ for each $k = 0,1,2,3,4$.

The boundary conditions in the problem (4.1) are reduced to pure anti-periodic boundary conditions by choosing $\theta_k = -1$ and $\beta_k = 0$ for each $k = 0,1,2,3,4$. On the other hand, the boundary conditions of (4.1) for the cases $\theta_k = 0$ can be thought as an extension of nonlocal integral boundary valued problem.

This section gives some basic concepts of multivalued maps [28] on a normed space X.

A multivalued function (see [32]) is a function that assumes two or more distinct values in its range for at least one point in its domain. While these functions are not functions in the normal sense of being one-to-one or many-to-one, the usage is so common that there is no way to dislodge it. When considering multivalued functions, it is therefore necessary to refer to usual ”functions” as single-valued functions.

Definition 4.1 *[28]A mapping $G : X \to Y$ associating with each element x of a set X, a subset $G(x)$ of a set Y.*

Denote $\mathscr{P}(X)$ is the family of all subsets of X, $P_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}$, $P_{cp,c}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}$.

Definition 4.2 *[28]A multivalued map $G : X \to \mathscr{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.*

Definition 4.3 *[28]A multivalued map $G : X \to \mathscr{P}(X)$ is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$.*

Definition 4.4 *[28]A multivalued map $G : X \to \mathscr{P}(X)$ is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.*

Definition 4.5 *[28]A multivalued map $G : X \to \mathscr{P}(X)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$, or equivalently: If for every open set $U \subset \mathscr{P}(X)$, the set $\{x \in X : G(x) \cap U \neq \phi\}$ is open in X.*

Theorem 4.6 *I[28]f the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.*

Definition 4.7 *[28]G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by Fix G.*

Definition 4.8 *[28]A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y,G(t)) = inf\{|y-z| : z \in G(t)\}$ is measurable.*

Let $L^1(J,\mathbb{R})$ be the Banach space of all measurable functions $x : J \to \mathbb{R}$ which are Lebesque integrable endowed with the norm

$$\|x\|_{L^1} = \int_{t_0}^{T} |x(t)|dt.$$

Definition 4.9 *[28]A multivalued map $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if the mapping $t \to F(t,x)$ is measurable for each $x \in \mathbb{R}$, and the mapping $x \to F(t,x)$ is upper semi-continuous for almost all $t \in J$.*

Definition 4.10 *[28]A Carathéodory function F is called L^1-Carathéodory if for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J,\mathbb{R}^+)$ such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in J$.

Definition 4.11 *[28]Let Y be a Banach space, Z a nonempty closed subset of Y and $G : Z \to \mathscr{P}(Y)$ a multivalued operator with nonempty closed values. G is called lower semi-continuous (l.s.c.) if the set $\{z \in Z : G(z) \cap B \neq \phi\}$ is open for any open set B in Y.*

Definition 4.12 *[28]If a multivalued mapping is both lower semi-continuous and upper semi-continuous, then it is called a continuous multivalued mapping.*

Theorem 4.13 *T[28]he Cartesian product of a finite family of upper semi-continuous multivalued mappings is upper semi-continuous.*

Definition 4.14 *[28]Let A be a subset of $J \times \mathbb{R}$. A is $\ell \otimes B$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathscr{F} \times \mathscr{L}$, where $\mathscr{F}$ is Lebesque measurable in J and $\mathscr{L}$ is Borel measurable in $\mathbb{R}$.*

Definition 4.15 *[28]A subset A of $L^1(J,\mathbb{R})$ is decomposable if for all $u,v \in A$ and a measurable subinterval $I \subset J$, the function $u\chi_I + v\chi_{J-I} \in A$, where χ_I stands for the characteristic function of I.*

Definition 4.16 *[28]If for a multivalued mapping $F : X \to Y$, there exists a continuous single valued mapping $f : X \to Y$ with $f(x) \in F(x)$ for every $x \in X$, we say that f is a selection of F and that F is selectionable.*

Definition 4.17 *[28]If $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map with nonempty compact values and $u \in C(J,\mathbb{R})$, then the set of selections of $F(.,.)$, denoted by $S_{F,u}$, is of lower semi-continuous type if $S_{F,u} = \{w \in L^1(J,\mathbb{R}) : w(t) \in F(t,u(t))$ for a.e. $t \in J\}$ is lower semi-continuous with nonempty closed and decomposable values.*

Theorem 4.18 *[28]If $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multivalued map is such that $F(.,x) : J \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$, then F has a measurable selection.*

Definition 4.19 *[28]Let (X,d) be a metric space associated with the norm $\|\cdot\|$. The Pompeiu-Hausdorff distance of the closed subsets $A,\ B \subset\ X$ is defined by*

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, d^*(A,B) = \sup\{d(a,B) : a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

Definition 4.20 *[28]A multivalued operator N on X with nonempty values in X is called*

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(N(x),N(y)) \leq \gamma d(x,y), \text{for each } x,y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following results will be used in what follows.

Lemma 4.21 *[28]Let X be a Banach space, $F : J \times X \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to P_{cp,c}(C(J,X)), x \to (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Theorem 4.22 *([12]) Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1(J,\mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $N(.)$ has a continuous selection, i.e., there exists a continuous mapping (single valued) $g : Y \to L^1(J,\mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.*

Theorem 4.23 *Let $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
(a) $(t,x) \to F(t,x)$ is $\ell \otimes B$ measurable,
(b) $x \to F(t,x)$ is lower semi-continuous for each $t \in J$,
(c) there exists a continuous nondecreasing function $\psi : [t_0,\ \infty) \to (0,\ \infty)$ and a function $p \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq p(t)\psi(|x|) \ \text{for each}\ (t,x) \in J \times \mathbb{R}.$$

Then F has a lower semi continuous property.

The following fixed point theorem for multivalued contraction mappings appears in Covitz and Nadler [16].

Theorem 4.24 *[16]Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \phi$.*

The Leray-Schauder nonlinear alternative for compact single-valued mappings is given by the following theorem.

Theorem 4.25 *[16]Let E be a Banach space, C be a closed convex subset of E, U be an open subset of C, and $0 \in U$. Suppose that $G : \overline{U} \to \mathscr{P}(C(J,\mathbb{R}))$ is an upper semicontinuous compact map. Then either G has a fixed point in $\overline{U}$, or there is an $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

4.2 The Equivalent Integral Form of the Anti- Periodic Boundary Value Problem

The equivalent integral form of the corresponding linear anti-periodic fractional differential equation of (4.1) with $\beta_k = 0, k = 0,1,2,3,4$, is obtained in terms of Green's functions. We assume in this section that $t_0 = 0$, and $J_0 = [0,T]$.

Lemma 4.26 *Let $y \in C(J_0, \mathbb{R})$. Then the solution of the anti-periodic boundary value problem*

$$\begin{cases} {}^cD_0^q x(t) = y(t), t \in J_0, 4 < q \leq 5, \\ x^{(k)}(0) = -x^{(k)}(T), k = 0,1,2,3,4, \end{cases} \tag{4.2}$$

is

$$x(t) = \int_0^T G(t,s)\, y(s)\, ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \begin{cases} \frac{2(t-s)^{q-1}-(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} + \\ \frac{\left(6t^2T-4t^3-T^3\right)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{\left(3t^3T-t^4-tT^3\right)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 \leq s < t \leq T, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q--1)} + \frac{t(T-t)(T-s)^{q-2}}{4\Gamma(q-2)} + \\ \frac{\left(6t^2T-4t^3-T^3\right)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{\left(3t^3T-t^4-tT^3\right)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 \leq t < s \leq T. \end{cases}$$

Proof. Using Lemma 2.6, for some constants $c_1, c_2.c_3, c_4, c_5 \in \mathbb{R}$, we have

$$\begin{aligned} x(t) &= I_0^q y(t) - c_1 - c_2 t - c_3 t^2 - c_4 t^3 - c_5 t^4 \\ &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds - c_1 - c_2 t - c_3 t^2 - c_4 t^3 - c_5 t^4. \end{aligned} \tag{4.3}$$

Applying the boundary conditions for problem (4.2) in (4.3), we find that

$$c_1 = \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)\, ds - \frac{T}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\, ds + \frac{T^3}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\, ds,$$

$$c_2 = \frac{1}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\, ds - \frac{T}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\, ds + \frac{T^3}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\, ds,$$

$$c_3 = \frac{1}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\, ds - \frac{T}{8}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\, ds,$$

$$c_4 = \frac{1}{12}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\, ds - \frac{T}{24}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\, ds,$$

and

$$c_5 = \frac{1}{48}\int_0^T \frac{(T - s)^{q-5}}{\Gamma(q-4)} y(s)\, ds.$$

Substituting the values of c_1, c_2, c_3, c_4 and c_5 in (4.3), we obtain

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)\,ds \\ &\quad + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds \\ &\quad + \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\,ds \\ &\quad + \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\,ds \\ &= \int_0^T G(t,s)\,y(s)\,ds, \end{aligned}$$

which completes the proof. ■

Observe that

$$\begin{cases} (T-t)(T-s)^{q-5} \le (T-t)^{q-4}, t < s, \\ (T-t)(T-s)^{q-5} \le (T-s)^{q-4}, t \ge s. \end{cases} \tag{4.4}$$

4.3 Existence of Solutions For Fractional Differential Inclusions with Anti-periodic Boundary Conditions

We obtain the existence of solution to fractional inclusion (4.1), in the case of $\theta_k = -1$, and $\beta_k = 0$, for all $k = 0,1,2,3,4$, that is, the equation

$$\begin{cases} {}^cD_0^q x(t) \in F(t, x(t)), t \in J_0,\ q \in (4,5] \\ x^{(k)}(0) = -x^{(k)}(T),\ k = 0,1,2,3,4. \end{cases} \tag{4.5}$$

where $F : J_0 \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, and $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$. We introduce next, the first existence result of (4.5) using the nonlinear alternative of the Leray-Schauder fixed point theorem.

Theorem 4.27 *Assume that*

(H1) $F : J_0 \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is Carathéodory and has convex values,*

(H2) *there exists a continuous nondecreasing function* $\psi : [0,\ \infty) \to (0,\ \infty)$ *and a function* $p \in L^1(J_0, \mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq p(t)\psi(|x|) \text{ for each } (t,x) \in J_0 \times \mathbb{R},$$

(H3) *there exists a number* $M > 0$ *such that* $\frac{M}{\gamma\psi(M)\|p\|_{L^1}} > 1$, *where*

$$\gamma = \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2} + \frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right).$$

Then the boundary value problem (4.5) has at least one solution on J_0.

Proof. Define an operator

$$\begin{aligned}\Omega(x) &= \Bigg\{h \in C(J_0,\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\,ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\,ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s)\,ds \Bigg\}\end{aligned}$$

for $f \in S_{F,x}$. We show that Ω satisfies the assumptions of the nonlinear alternative of the Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C(J_0,\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$, then there exist $f_1,\ f_2 \in S_{F,x}$ such that for each $t \in J_0, i = 1,2$, we have

$$\begin{aligned}h_i(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)\,ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f_i(s)ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f_i(s)ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f_i(s)ds.\end{aligned}$$

Let $0 \leq \omega \leq 1$, then, for each $t \in J_0$, we have

$$\begin{aligned}
[\omega h_1 + (1-\omega)h_2](t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
&-\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
&+\frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
&+\frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}[\omega f_1(s) + (1-\omega) f_2(s)]ds + \\
&\frac{\left(6t^2T - 4t^3 - T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\
&+\frac{\left(2t^3T - t^4 - tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}[\omega f_1(s) + (1-\omega) f_2(s)]ds.
\end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$. Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C(J_0,\mathbb{R})$. For a positive number r, let $B_r = \{x \in C(J_0,\mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C(J_0,\mathbb{R})$. Then, for each $h \in \Omega(x),\ x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned}
h(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\
&+\frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\,ds \\
&+\frac{\left(6t^2T - 4t^3 - T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)ds \\
&+\frac{\left(2t^3T - t^4 - tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s)\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
|h(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s)|ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s)|ds \\
&\quad + \frac{|T-2t|}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s)|ds + \frac{|t(T-t)|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s)|ds \\
&\quad + \frac{|6t^2T-4t^3-T^3|}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}|f(s)|ds \\
&\quad + \frac{|2t^3T-t^4-tT^3|}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}|f(s)|ds \\
&\leq \frac{\psi(M)T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right)\int_0^T p(s)ds,
\end{aligned}$$

where we have used (H_2) and (H_3). Now we show that Ω maps bounded sets into equicontinuous sets of $C(J_0,\mathbb{R})$. Let $t_1,t_2 \in J_0$ with $t_1 < t_2$ and $x \in B_r$ where B_r is a bounded set of $C(J_0,\mathbb{R})$. In view of (H_3), For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq \int_0^{t_2} \frac{|(t_2-s)^{q-1}-(t_1-s)^{q-1}|}{\Gamma(q)}|f(s)|ds + \int_{t_1}^{t_2} \frac{|(t_1-s)^{q-1}|}{\Gamma(q)}|f(s)|ds \\
&\quad + \frac{|t_2-t_1|}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s)ds| \\
&\quad + \frac{|(t_2-t_1)(T-t_2-t_1)|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s)|ds \\
&\quad + \frac{|(t_2-t_1)\left[3T(t_2+t_1)-2(t_2^2+t_2t_1+t_1^2)\right]|}{24}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}|f(s)|ds \\
&\quad + \frac{|(t_2-t_1)[2(t_2^2+t_1t_2+t_1^2)T-(t_2+t_1)(t_2^2+t_1^2)-T^3]|}{48} \\
&\quad \times \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}|f(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_2} \frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}|}{\Gamma(q)} p(s)\psi(M)ds \\
&+\int_{t_1}^{t_2} \frac{|(t_2-s)^{q-1}|}{\Gamma(q)} p(s)\psi(M)ds \\
&+\frac{|t_2-t_1|}{2}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} p(s)\,\psi(M)\,ds \\
&+\frac{|(t_2-t_1)[T-t_2-t_1]|}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} p(s)\,\psi(M)\,ds \\
&+\frac{|(t_2-t_1)[3T(t_2+t_1)-2(t_2^2+t_2t_1+t_1^2)]|}{24} \\
&\times\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} p(s)\,\psi(M)\,ds \\
&+\frac{|(t_2-t_1)[2(t_2^2+t_2t_1+t_1^2)T-(t_2+t_1)(t_2^2+t_1^2)-T^3]|}{48} \\
&\times\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} p(s)\psi(M)ds.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω satisfies the above three assumptions, it follows by the Arzela–Ascoli theorem that $\Omega : C(J_0,\mathbb{R}) \to \mathscr{P}(C(J_0,\mathbb{R}))$ is completely continuous. In our next step, we show that Ω has a closed graph. Let $x_n \to x_*$, $h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$, such that for each $t \in J_0$, we have

$$
\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_n(s)ds \\
&+\frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_n(s)ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f_n(s)ds \\
&+\frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f_n(s)ds \\
&+\frac{(2t^3T-t^4-tT^3)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f_n(s)ds.
\end{aligned}
$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in J_0$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f_*(s)ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f_*(s)ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f_*(s)ds. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1(J_0,\mathbb{R}) \to C(J_0,\mathbb{R})$ given by

$$\begin{aligned} f \quad \to \quad & \Theta(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ & + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)\,ds \\ & + \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)\,ds \\ & + \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s)\,ds. \end{aligned}$$

Observe that

$$\begin{aligned}
|h_n(t)-h_*(t)| &= \Bigg|\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))\,ds \\
&\quad -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))\,ds \\
&\quad +\frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}(f_n(s)-f_*(s))\,ds \\
&\quad +\frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}(f_n(s)-f_*(s))\,ds \\
&\quad +\frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}(f_n(s)-f_*(s))\,ds \\
&\quad +\frac{\left(3t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}(f_n(s)-f_*(s))ds\Bigg|
\end{aligned}$$

which converges to 0, as $n\to\infty$. Thus, it follows by Lemma 4.21 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x})$, since $x_n\to x_*$, we have

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds-\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f_*(s)ds \\
&\quad +\frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f_*(s)ds \\
&\quad +\frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}f_*(s)ds \\
&\quad +\frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}f_*(s)ds \\
&\quad +\frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}f_*(s)ds.
\end{aligned}$$

for some $f_* \in S_{F,x_*}$. Finally, we discuss a priori bounds on solutions. Let x be a solution of (4.5). Then there exists $f \in L^1(J_0,\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in J_0$, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s)ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s)ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s)ds. \end{aligned}$$

For each $t \in J_0$, using $(H2)$, we obtain

$$\begin{aligned} |x(t)| &\leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right)\int_0^T f(s)ds \\ &\leq \gamma\psi(\|x\|)\int_0^T p(s)ds. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\gamma\psi(\|x\|)\,\|p\|_{L^1}} \leq 1.$$

In view of $(H3)$, there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega : \overline{U} \to \mathscr{P}(C(J_0,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no

$$x \in \partial U$$

such that

$$x \in \mu\Omega(x)$$

for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of the Leray-Schauder type, we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of problem (4.5). This completes the proof. ■

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of the Leray-Schauder type together with the selection theorem of Bressan and Colombo [12] for lower semi-continuous maps with decomposable values.

Theorem 4.28 *Assume that (H2), (H3) hold. Moreover, Assume the following:*

(H4) $F : J_0 \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \to F(t,x)$ *is* $\ell \otimes B$ *measurable,*

(b) $x \to F(t,x)$ *is lower semi-continuous for each* $t \in J_0$*,*

(H5) *For each* $\sigma > 0$*, there exists* $\varphi_\sigma \in L^1(J_0,\mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\sigma(t) \text{ for all } \|x\| \leq \sigma \text{ and for a.e. } t \in J_0.$$

Then the boundary value problem (4.5) has at least one solution on J_0*.*

Proof. It follows from $(H4)$ and $(H5)$ that F is of $l.s.c.$ type. Then from Theorem 4.22, there exists a continuous function $f : C(J_0,\mathbb{R}) \to L^1(J_0,\mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in C(J_0,\mathbb{R})$. Consider the problem

$$\begin{cases} {}^cD_0^q x(t) = f(x(t)), t \in J_0, q \in (4,5] \\ x^{(k)}(0) = -x^{(k)}(T), k = 0,1,2,3,4. \end{cases} \tag{4.6}$$

Observe that if $x \in C(J_0,\mathbb{R})$ is a solution of (4.6), then x is a solution to problem (4.5). In order to transform problem (4.6) into a fixed point problem, we define the operator Π as

$$
\begin{aligned}
\Pi x(t) \;=\; & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\
& + \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(x(s))ds \\
& + \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(x(s))ds \\
& + \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(x(s))ds \\
& + \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(x(s))ds.
\end{aligned}
$$

It can easily be shown that Π is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.27, so we omit it. This completes the proof. ■

Next, we show the existence of solutions for problem (4.5) with a nonconvex value by applying a fixed point theorem of multivalued maps due to Covitz and Nadler [16].

Theorem 4.29 *Assume that the following conditions hold:*

(H6) $F: J_0 \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x): J_0 \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$,

(H7) $d_H(F(t,x),F(t\ ,y)) \le z(t)|x-y|$ *for almost all* $t \in J_0$ *and* $x,y \in \mathbb{R}$ *with*

$$z \in L^1(J_0,\mathbb{R}^+) \text{ and } d(0,F(t,0)) \le z(t) \text{ for almost all } t \in J_0.$$

Then the boundary value problem (4.5) has at least one solution on J if

$$\frac{T^{q-1}\|z\|_{L^1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right) < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J_0,\mathbb{R})$ by assumption $(H6)$, so F has a measurable selection (see [28]: Theorem 3.6). Now we show that the operator Ω satisfies the assumptions of Theorem 4.24. To show that $\Omega(x) \in P_{cl}((C(J_0,\mathbb{R}))$ for each $x \in C(J_0,\mathbb{R})$, let

$(u_n)_{n\geq 0} \in \Omega(x)$ be such that $u_n \to u$ in $C(J_0,\mathbb{R})$. Then $u \in C(J_0,\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in J_0$,

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)\,ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v_n(s)ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v_n(s)ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} v_n(s)ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J_0,\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J_0$,

$$\begin{aligned} u_n(t) &\to u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(t)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(t)\,ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(t)\,ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v(t)\,ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v(t)\,ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} v(t)\,ds. \end{aligned}$$

Hence, $u \in \Omega(x)$. Next, we show that there exists $\tau < 1$ such that

$$d_H(\Omega(x),\Omega(y)) \leq \tau \left\|x-y\right\|,$$

for each $x,y \in C(J_0,\mathbb{R})$. Let $x,y \in C(J_0,\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in J_0$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_1(s)\,ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v_1(s)ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v_1(s)ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} v_1(s)ds. \end{aligned}$$

By $(H7)$, we have

$$d_H(F(t,x),F(t,y)) \le z(t)|x(t)-y(t)|.$$

So, there exists $w \in F(t,y(t))$ such that

$$|v_1(t)-w| \le z(t)|x(t)-y(t)|,\ t \in J.$$

Define $V : J_0 \to \mathscr{P}(\mathbb{R})$ by

$$V(t) = \{w \in \mathbb{R} : |v_1(t)-w| \le z(t)\,|x(t)-y(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t,y(t))$ is measurable ([28]: Proposition 3.4), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t,y(t))$ and for each $t \in J_0$, we have $|v_1(t)-v_2(t)| \leq z(t)|x(t)-y(t)|$. For each $t \in J_0$, let us define

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_2(s)\,ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_2(s)\,ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} v_2(s)\,ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} v_2(s)\,ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} v_2(s)\,ds. \end{aligned}$$

Thus, for each $t \in J_0$, it follows that

$$\begin{aligned} |h_1(t)-h_2(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s)-v_2(s)|ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |v_1(s)-v_2(s)|ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |v_1(s)-v_2(s)|ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |v_1(s)-v_2(s)|ds \\ &+ \frac{\left(6t^2T-4t^3-T^3\right)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |v_1(s)-v_2(s)|ds \\ &+ \frac{\left(2t^3T-t^4-tT^3\right)}{48}\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} |v_1(s)-v_2(s)|ds \\ &\leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right) \\ &\times \int_0^T z(s)\,\|x-y\|ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|h_1-h_2\| &\leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right) \\ &\times \|z\|_{L^1}\|x-y\|. \end{aligned}$$

We deduce that

$$\begin{aligned} d_H(\Omega(x),\Omega(y)) &\leq \frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right)\|z\|_{L^1}\|x-y\| \\ &\leq \tau\|x-y\|. \end{aligned}$$

Since Ω is a contraction, it follows by Theorem 4.24 that Ω has a fixed point x which is a solution of (4.5). This completes the proof. ■

The new existence results for a class of fifth-order nonlinear differential inclusions with anti-periodic boundary conditions follow as a special case by taking $q = 5$ in the results of this section.

4.4 Examples of Anti-periodic Fractional Differential Inclusions

Example 4.30 *Consider the following fractional differential inclusion*

$$\begin{cases} {}^cD_0^{4.2}x(t) \in F(t,x(t)), t \in [0,1], \\ x^{(k)}(0) = -x^{(k)}(1), k = 0,1,2,3,4. \end{cases} \tag{4.7}$$

where $F : [0,1]\times\mathbb{R}\to\mathbb{R}$ *is a multivalued map given by*

$$F(t,x) = \left\{y\in\mathbb{R} : e^{-|x|}+\sin t+t^2 \leq y \leq 2+\frac{|x|}{1+x^2}+t^3\right\}.$$

Observe that

$$t\to F(t,x) = \left\{y\in\mathbb{R} : e^{-|x|}+\sin t+t^2 \leq y \leq 2+\frac{|x|}{1+x^2}+t^3\right\}$$

is measurable for each $x\in\mathbb{R}$, *since*

$$y = e^{-|x|}+\sin t+t^2$$

and

$$y = 2+\frac{|x|}{1+x^2}+t^3$$

are measurable on $[0,1]\times\mathbb{R}$.

Now let (t_0,x_0) *be any element in* $[0,1]\times\mathbb{R}$.

Then

$$F(t_0,x_0) = \left\{y\in\mathbb{R} : e^{-|x_0|}+\sin t_0+t_0^2 \leq y \leq 2+\frac{|x_0|}{1+x_0^2}+t_0^3\right\}$$

is a closed subset of $[0,1]\times\mathbb{R}$ *and*

$$\left\{y\in\mathbb{R}: e^{-|x_0|}+\sin t_0+t_0^2\le y\le 2+\frac{|x_0|}{1+x_0^2}+t_0^3\right\}\neq\phi.$$

Let O *be any open interval in* $\mathbb{R}^+$ *such that*

$$\left\{y\in\mathbb{R}: e^{-|x_0|}+\sin t_0+t_0^2\le y\le 2+\frac{|x_0|}{1+x_0^2}+t_0^3\right\}\subset O,$$

we can find an open intervals U *of* t_0 *and* V *of* x_0 *with* $[0,1]\subset U=(-\varepsilon,1+\varepsilon)$ *and* $F(u,v)\subset O$ *for every* $u\in U$ *and* $v\in V$, *where* ε *is a small positive real number. So* $x\to F(t,x)$ *is upper semi-continuous for all* $t\in J$. *Thus* F *is a Carathéodory and clearly has convex values satisfying*

$$\|F(t,x)\|=\sup\{|y|: y\in F(t,x)\}\le 4 \text{ for each } (t,x)\in[0,1]\times\mathbb{R},$$

with $p(t)=1$, *and* $\psi(\|x\|)=4$. *Furthermore, let* M *be any number satisfying*

$$\begin{aligned} M &> \frac{T^{q-1}\psi(M)\,\|p\|_{L^1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(2q^3-13q^2+39q-30)}{48}\right)\\ &> 3.49. \end{aligned}$$

Clearly, all the conditions of Theorem 4.27 are satisfied. So there exists at least one solution of problem (4.7) on $[0,1]$.

Example 4.31 *Consider the following fractional differential inclusion*

$$\begin{cases} {}^cD_0^{\frac{19}{4}}x(t)\in F(t,x(t)),t\in[0,1],\\ x^{(k)}(0)=-x^{(k)}(1),k=0,1,2,3,4, \end{cases}\tag{4.8}$$

where $F:[0,1]\times[0,\frac{\pi}{2}]\to\mathbb{R}^+$ *is a multivalued map given by*

$$F(t,x)=\left[0,\frac{\sin x}{(2+t)^4}\right].$$

Now

$$\begin{aligned} \sup\{|y| &: y\in F(t,x)\}\\ &\le \frac{\sin x}{(2+t)^4}\\ &\le \frac{1}{16} \text{ for each } (t,x)\in[0,1]\times\left[0,\frac{\pi}{2}\right], \end{aligned}$$

and

$$\begin{aligned}
d_H(F(t,x),F(t,y)) &= d_H\left(\left[0,\frac{\sin x}{(2+t)^4}\right],\left[0,\frac{\sin y}{(2+t)^4}\right]\right)\\
&= \max\left\{d^*\left(\left[0,\frac{\sin x}{(2+t)^4}\right],\left[0,\frac{\sin y}{(2+t)^4}\right]\right)\right.\\
&\quad \left.,d^*\left(\left[0,\frac{\sin y}{(2+t)^4}\right],\left[0,\frac{\sin x}{(2+t)^4}\right]\right)\right\}\\
&= \max\left\{\sup\left\{d\left(a,\left[0,\frac{\sin y}{(2+t)^4}\right]\right):a\in\left[0,\frac{\sin x}{(2+t)^4}\right]\right\},\right.\\
&\quad \left.\sup\left\{d\left(\left[0,\frac{\sin x}{(2+t)^4}\right],b\right):b\in\left[0,\frac{\sin y}{(2+t)^4}\right]\right\}\right\}\\
&\leq \frac{1}{(2+t)^4}|x-y|.
\end{aligned}$$

Here $z(t)=\frac{1}{(2+t)^4}$, *with* $\|z\|_{L^1}=0.017$, *and*

$$\frac{T^{q-1}}{\Gamma(q)}\left(\frac{3}{2}+\frac{(q-1)(3q^3-22q^2+65q-54)}{48}\right)\|z\|_{L^1}<1.$$

The compactness of F *together with the above calculations lead to the existence of solution of the problem (4.8) by Theorem 4.29.*

4.5 Existence of Solution For Fractional Differential Inclusions with Anti-periodic and Integral Type Boundary Conditions

We establish, in this section, the existence of solution for the fractional inclusion (4.1) that is a generalization of the fractional inclusion (4.5). Hence, assume that $J=[t_0,T]$, and $\lambda_k=max_{t\in J}|\lambda_k(t)|$

Theorem 4.32 *Assume that*

(HA) $F:J\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ *is Carathéodory and has convex values,*

(HB) *there exists a continuous nondecreasing function* $\psi:[t_0,\infty)\to(0,\infty)$ *and a function* $p\in L^1(J,\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|=\sup\{|v|:v\in F(t,x)\}\leq p(t)\psi(|x|),$$

for each $(t,x)\in J\times\mathbb{R}$,

(HC) *there exist continuous nondecreasing functions* $\psi_k : [t_0,\infty) \to (0,\infty)$ *and functions* $p_k \in L^1(J,\mathbb{R}^+)$ *such that*

$$|g_k(t,x)| \le p_k(t)\psi_k(|x|), k = 0,1,2,3,4,$$

for each $(t,x) \in J \times \mathbb{R}$,

(HD) *there exists a large number* $M > 0$ *such that*

$$\frac{M}{\gamma_1 \psi(M) \|p\|_{L^1} + \gamma_2} > 1,$$

where

$$\gamma_1 = \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{|\theta_k| \lambda_k (T-t_0)^{q-k-1}}{k!\Gamma(q-k)|\alpha_k|}$$

and

$$\gamma_2 = \sum_{k=0}^{4} \frac{\lambda_k |\beta_k|}{k! |\alpha_k|} \psi_k(M) \|p_k\|_{L^1}.$$

Then the boundary value problem (4.1) has at least one solution on J.

Proof. Define an operator

$$\begin{aligned} \Omega(x) \;=\; & \Bigg\{ h \in C(J,\mathbb{R}) : h(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ & + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\ & + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds \Bigg\} \end{aligned}$$

for $f \in S_{F,x}$. We show that Ω satisfies the assumptions of the nonlinear alternative of the Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit its proof. In the second step, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C(J,\mathbb{R})$.

Then, for each $h \in \Omega(x),\ x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\ &\quad + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

Then for $t \in J$, we have

$$\begin{aligned} |h(t)| &\leq \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)|\,ds + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s)|\,ds \\ &\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))|\,ds \\ &\leq \psi(\|x\|)\,\|p\|_{L^1} \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k\,|\theta_k|\,(T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|} \right\} \\ &\quad + \sum_{k=0}^{4} \frac{|\beta_k|\,\lambda_k}{k!\,|\alpha_k|} \psi_k(\|x\|)\,\|p_k\|_{L^1}. \end{aligned}$$

Thus,

$$\begin{aligned} \|h\| &\leq \psi(\|r\|)\,\|p\|_{L^1} \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k\,|\theta_k|\,(T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|} \right\} \\ &\quad + \sum_{k=0}^{4} \frac{\lambda_k\,|\beta_k|}{k!\,|\alpha_k|} \psi_k(\|r\|)\,\|p_k\|_{L^1}. \end{aligned}$$

Now we show that Ω maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, and $x \in B_r$ where B_r is a bounded set of $C(J,\mathbb{R})$. In view of (H_3), for each $h \in \Omega(x)$, we obtain

$$\begin{aligned} &|h(t_2)-h(t_1)| \\ = &\left| \int_{t_0}^{t_2} \frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s)\,ds \right. \\ &+ \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} (\lambda_k(t_2)-\lambda_k(t_1)) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s)\,ds \\ &\left. + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} (\lambda_k(t_2)-\lambda_k(t_1)) \int_{t_0}^{T} g_k(s,x(s))ds \right| \end{aligned}$$

$$\leq \psi(\|x\|)\int_{t_0}^{t_2}\frac{\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|}{\Gamma(q)}p(s)ds$$

$$+\psi(\|x\|)\int_{t_1}^{t_2}\frac{|(t_2-s)^{q-1}|}{\Gamma(q)}p(s)ds$$

$$+\sum_{k=0}^{4}\frac{|\theta_k|}{k!\,|\alpha_k|}\left|\lambda_k(t_2)-\lambda_k(t_1)\right|\psi(\|x\|)\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}p(s)ds$$

$$+\sum_{k=0}^{4}\frac{|\beta_k|}{k!\,|\alpha_k|}\left|\lambda_k(t_2)-\lambda_k(t_1)\right|\psi_k(\|x\|)\int_{t_0}^{T}p_k(s)ds$$

The functions $\lambda_k(t)$ behave like a polynomial function, particularly, $(\lambda_k(t_2)-\lambda_k(t_1))\to 0$, as $t_2-t_1\to 0$. Therefore, the right hand side of the above inequality tends to zero independently of $x\in B_r$ as $t_2-t_1\to 0$. As Ω satisfies the above three assumptions, it follows by the Arzela-Ascoli theorem that $\Omega: C(J,\mathbb{R})\to\mathscr{P}(C(J,\mathbb{R}))$ is completely continuous. In our next step, we show that Ω has a closed graph. Let $x_n\to x_*$, $h_n\in\Omega(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega(x_*)$. Associated with $h_n\in\Omega(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in J$,

$$h_n(t) = \int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f_n(s)\,ds+\sum_{k=0}^{4}\frac{\theta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}f_n(s)\,ds$$
$$+\sum_{k=0}^{4}\frac{\beta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T}g_k(s,x_n(s))ds.$$

Thus we have to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in J$,

$$h_*(t) = \int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)\,ds+\sum_{k=0}^{4}\frac{\theta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}f_*(s)\,ds$$
$$+\sum_{k=0}^{4}\frac{\beta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T}g_k(s,x_*(s))ds. \tag{4.9}$$

Let us consider the continuous linear operator $\Theta: L^1(J,\mathbb{R})\to C(J,\mathbb{R})$ given by

$$f \to \Theta(f)(t)=\int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f(s)\,ds+\sum_{k=0}^{4}\frac{\theta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}f(s)\,ds$$
$$+\sum_{k=0}^{4}\frac{\beta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T}g_k(s,x(s))ds.$$

Observe that

$$|h_n(t)-h_*(t)| \leq \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f_n(s)-f_*(s)| ds + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f_n(s)-f_*(s)| ds + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x_n(s))-g_k(s,x_*(s))| ds.$$

Thus, it follows by Lemma 4.21 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x})$, since $x_n \to x_*$, then h_* satisfying (4.9) for some $f_* \in S_{F,x_*}$. Finally, we discuss a priori bounds on solutions. Let x be a solution of (4.1), then there exists $f \in L^1(J,\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s) ds + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s)) ds.$$

Using the computations of the second step above, we have

$$\begin{aligned} |x(t)| &\leq \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)| ds + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s)| ds \\ &\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))| ds \\ &\leq \psi(\|x\|) \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k |\theta_k| (T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|} \right\} \int_{t_0}^{T} p(s) ds \\ &\quad + \sum_{k=0}^{4} \frac{\lambda_k |\beta_k|}{k!\,|\alpha_k|} \psi_k(\|x\|) \int_{t_0}^{T} p_k(s) ds. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\gamma_1 \psi(\|x\|) \|p\|_{L^1} + \gamma_2} \leq 1.$$

In view of $(H4)$, there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega : \overline{U} \to \mathscr{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of the Leray–Schauder type, we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of problem (4.1). This completes the proof. ■

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of the Leray–Schauder type together with the selection theorem of Bressan and Colombo [12] for lower semi-continuous maps with decomposable values.

Theorem 4.33 *Assume that (HB), (HC), (HD), and the following condition holds*

(HE) $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \to F(t,x)$ *is* $\ell \otimes B$ *measurable,*

(b) $x \to F(t,x)$ *is lower semi-continuous for each* $t \in J$.

Then the boundary value problem (4.1) has at least one solution on J.

Proof. It follows from (HB) and (HE) that F is of $l.s.c.$ type. Then from Theorem 4.22, there exists a continuous function $f : C(J,\mathbb{R}) \to L_1(J,\mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in C(J,\mathbb{R})$. Consider the problem

$$\begin{cases} {}^cD^q_{t_0}x(t) = f(x(t)), t \in J, q \in (4,5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t,x(t))dt, k = 0,1,2,3,4. \end{cases} \tag{4.10}$$

Observe that if $x \in C(J,\mathbb{R})$ is a solution of (4.10), then x is a solution to problem (4.1). In order to transform problem (4.10) into a fixed point problem, we define the operator Π as

$$\begin{aligned} \Pi x(t) \quad = \quad & \int\limits_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int\limits_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(x(s))\,ds \\ & + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int\limits_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

It can easily be shown that Π is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.32, so we omit it. This completes the proof. ■

Now we prove the existence of solutions for problem (4.1) with a nonconvex value by applying a fixed point theorem for a multivalued map due to Covitz and Nadler [16].

Theorem 4.34 *Assume that the following conditions hold:*

(HF) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) : J \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$,

(HG) $d_H(F(t,x), F(t\ ,y)) \leq z(t)|x-y|$ *for almost all* $t \in J$ *and* $x, y \in \mathbb{R}$ *with* $z \in L^1(J, \mathbb{R}^+)$ *and* $d(0, F(t,0)) \leq z(t)$ *for almost all* $t \in J$.

(HH) *There exist functions* $p_k \in L^1(J, \mathbb{R}^+)$ *such that*

$$|g_k(t,x) - g_k(t,y)| \leq p_k(t)|x-y|,$$

for $t \in J$, $k = 0,1,2,3,4$, *and* $x, y \in \mathbb{R}$.

Then the boundary value problem (4.1) has at least one solution on J if

$$\gamma_1 \|z\|_{L^1} + \omega < 1,$$

where

$$\omega = \sum_{k=0}^{4} \frac{\lambda_k \|z_k\|_{L^1} |\beta_k|}{k! |\alpha_k|}.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J, \mathbb{R})$ by assumption (HF), so F has a measurable selection (see [28]: Theorem 3.6). Now we show that the operator Ω satisfies the assumptions of Theorem 4.24. To show that $\Omega(x) \in P_{cl}((C(J, \mathbb{R}))$ for each $x \in C(J, \mathbb{R})$, let $(u_n)_{n \geq 0} \in \Omega(x)$ be such that $u_n \to u$ in $C\ (J, \mathbb{R})$. Then $u \in C(J, \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in J$,

$$\begin{aligned} u_n(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s) ds + \sum_{k=0}^{4} \frac{\theta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_n(s) ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k! \alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s, x(s)) ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$,

$$\begin{aligned} u_n(t) \quad \rightarrow \quad u(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v(s)ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

Hence, $u \in \Omega(x)$. Next we show that there exists $\tau < 1$ such that

$$d_H(\Omega(x),\Omega(y)) \leq \tau \|x-y\|,$$

for each $x,y \in C(J,\mathbb{R})$. Let $x,y \in C(J,\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists

$$v_1(t) \in F(t,x(t)),$$

such that, for each $t \in J$,

$$\begin{aligned} h_1(t) \quad = \quad & \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_1(s)ds \\ &+ \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

By (HG), we have

$$d_H(F(t,x),F(t,y)) \leq z(t)|x(t)-y(t)|.$$

So, there exists $w_* \in F(t,y(t))$ such that

$$|v_1(t)-w_*| \leq z(t)|x(t)-y(t)|,\ t \in J.$$

Define $V : J \to \mathscr{P}(\mathbb{R})$ by

$$V(t) = \{w_* \in \mathbb{R} : |v_1(t)-w_*| \leq z(t)\,|x(t)-y(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t,y(t))$ is measurable ([28]: Proposition 3.4), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t,y(t))$ and for each $t \in J$, we have $|v_1(t)-v_2(t)| \le z(t)|x(t)-y(t)|$. Define

$$\begin{aligned} h_2(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} v_2(s)ds \\ &\quad + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t)\int_{t_0}^{T} g_k(s,x(s))ds. \end{aligned}$$

Thus, for each $t \in J$, it follows that

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s)-v_2(s)|\,ds \\ &\quad + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |v_1(s)-v_2(s)|\,ds \\ &\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))-g_k(s,y(s))|\,ds \\ &\le \left\{ \left\{ \frac{(T-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\lambda_k\,|\theta_k|\,(T-t_0)^{q-k-1}}{k!\Gamma(q-k)\,|\alpha_k|} \right\} \|z\|_{L^1} \right. \\ &\quad \left. + \sum_{k=0}^{4} \frac{\lambda_k\,\|z_k\|_{L^1}\,|\beta_k|}{k!\,|\alpha_k|} \right\} \|x-y\|. \end{aligned}$$

Hence,

$$\|h_1-h_2\| \le (\gamma_1\,\|z\|_{L^1}+\omega)\,\|x-y\|.$$

We deduce that

$$\begin{aligned} d_H(\Omega(x),\Omega(y)) &\le (\gamma_1\,\|z\|_{L^1}+\omega)\,\|x-y\| \\ &\le \tau\|x-y\|. \end{aligned}$$

Since Ω is a contraction, it follows by Theorem 4.24 that Ω has a fixed point x which is a solution of (4.1). This completes the proof. ■

Remark 4.35 *The results for an anti-periodic boundary value problem (4.5) of fractional differential inclusions of order $q \in (4,5]$ follow as a special case by taking $\theta_k = -1$and $\beta_k = 0$ for each $k = 0,1,2,3,4$. In case of $\theta_k = 0$ for each $k = 0,1,2,3,4$, our results become an integral nonlocal* *boundary conditions given by*

$$x^{(k)}(t_0) = \beta_k \int_{t_0}^{T} g_k(t, x(t))dt, k = 0, 1, 2, 3, 4.$$

The new existence results for a class of fifth-order nonlinear differential inclusions with anti-periodic and integral boundary conditions follow as a special case by taking $q = 5$ in the results of this section.

On the other hand, several new results appear as a special case of the results obtained in this section by fixing the parameters involved in the problem (4.1).

Chapter 5

Conclusions

In this work we have studied the existence of solutions for fractional differential inclusions of order $q \in (4,5]$ with anti-periodic boundary conditions and anti-periodic type integral boundary conditions by applying some well known fixed point theorems and the Leray-Schauder degree theory and by means of some standard fixed point theorems for inclusions. Our results include the cases when the multivalued map involved in the problem has convex as well as non-convex values. Moreover, many nonlocal boundary value problems can be reduced to either anti-periodic or integral boundary conditions. The anti-periodic fractional differential equations that involved in the scientific researches were studied by using the technique of our problem such as ([13],[36],[37],[55],[58],[59],[61]) and devolve them with the regard of our introduced problem. However, theses theorems can be developed to more general approach in the theory of existence for fractional differential equations.

The fractional nonlinear differential equation

$$^{c}D_{t_0}^{q}x(t) = f(t,x(t)), t \in J = [t_0,T], T > t_0$$

of fractional order $q \in (5,6]$ can be considered by researchers together with anti-periodic and integral boundary conditions

$$x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^{T} g_k(t,x(t))dt, k = 0,1,2,3,4,5.$$

As well as, the corresponding fractional differential inclusion of the form

$$^{c}D_{t_0}^{q}x(t) \in F(t,x(t)), t \in J, q \in (5,6],$$

where $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, g_k, θ_k, and β_k are defined as above with anti-periodic type integral boundary condition.

More generally, one can consider the nonlinear differential equation

$$^{c}D_{t_0}^{q}x(t) = f(t, x(t)), t \in J = [t_0, T], T > t_0,$$

of arbitrary order $q \in (n-1, n]$ together with anti-periodic and integral boundary conditions

$$x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^{T} g_k(t, x(t))dt, k = 0, 1, 2, ..., n-1.$$

As well as, one can consider the correspondence of the above boundary value problems of arbitrary fractional in differential inclusion form

$$^{c}D_{t_0}^{q}x(t) \in F(t, x(t)), t \in J, q \in (n-1, n],$$

where $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, g_k, θ_k, and β_k are defined as above with anti-periodic type integral boundary condition.

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